The Dark Side of Circuit Breakers

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Abstract

Market-wide trading halts, also called circuit breakers, have been proposed and widely adopted as a measure to stabilize the stock market when experiencing large price movements. We develop an intertemporal equilibrium model to examine how circuit breakers impact the market when investors trade to share risk. We show that a downside circuit breaker tends to lower the stock price and increase its volatility, both conditional and realized. Due to this increase in volatility, the circuit breaker's own presence actually raises the likelihood of reaching the triggering price. In addition, the circuit breaker also increases the probability of hitting the triggering price as the stock price approaches it – the so-called "magnet effect." Surprisingly, the volatility amplification effect becomes stronger when the wealth share of the relatively pessimistic agent is small.

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1 Introduction

Large stock market swings in the absence of significant news often raise questions about the confidence in the market from market participants and policy makers alike. While the cause of these swings are not well understood, various measures have been proposed and adopted to halt trading during these extreme times in the hope to stabilize prices and maintain proper functioning of the market. These measures, sometimes referred to as throwing sand in the gears, range from market-wide trading halts, price limits on the whole market or individual assets, to limits on order flows and/or positions, and transaction taxes.[1] Yet, the merits of these measures, from a theoretical or an empirical perspective, remain largely unclear.

Probably one of the most prominent of these measures is the market-wide circuit breaker in the U.S., which was advocated by the Brady Commission (Presidential Task Force on Market Mechanisms, 1988) following the Black Monday of 1987 and subsequently implemented in 1988. It temporarily halts trading in all stocks and related derivatives when a designated market index drops by a significant amount. Following this lead, various forms of circuit breakers have been widely adopted by equity and derivative exchanges around the globe.[2]

Since its introduction, the U.S. circuit breaker was triggered only once on October 27, 1997 (see, e.g., Figure 1, left panel). At that time, the threshold was based on points movement of the DJIA index. At 2:36 p.m., a 350-point (4.54%) decline in the DJIA led to a 30-minute trading halt on stocks, equity options, and index futures. After trading resumed at 3:06 p.m., prices fell rapidly to reach the second-level 550-point circuit breaker point at 3:30 p.m., leading to the early market closure for the day.[3] But the market

[1]It is worth noting that contingent trading halts and price limits are part of the normal trading process for individual stocks or futures contracts. However, their presence there have quite different motivations. For example, the trading halt prior to large corporate announcement is motivated by the desire for fair information disclosure and daily price limits are motivated by the desire to guarantee the proper implementation of market to the market and deter market manipulation. In this paper, we focus on market-wide trading interventions in underlying markets such as stocks as well as their derivatives, which have very different motivations.

[2]According to a 2016 report, "Global Circuit Breaker Guide" by ITG, over 30 countries around the world have rules of trading halts in the form of circuit breakers, price limits and volatility auctions.

[3]For a detailed review of this event, see Securities and Exchange Commission (1998).

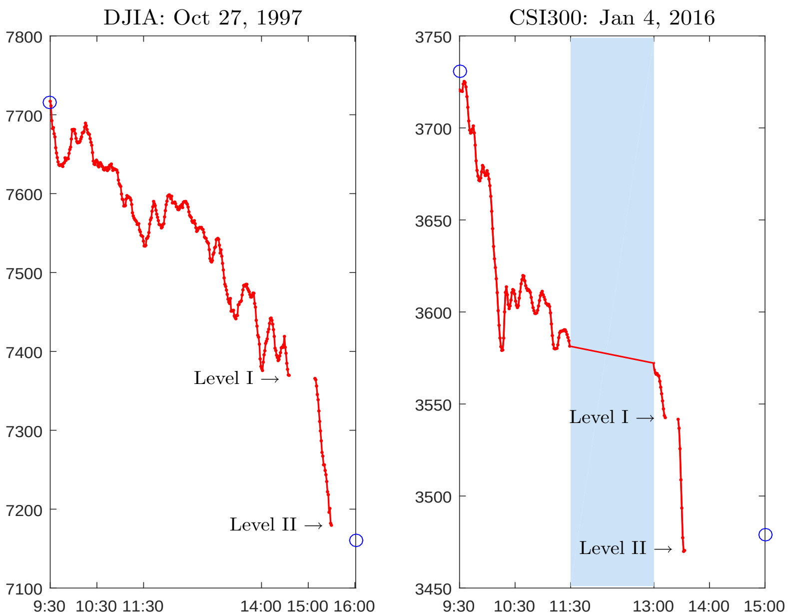

Figure 1: **Circuit breakers in the U.S. and Chinese stock market.** The left panel plots the DJIA index on Oct 27, 1997, when the market-wide circuit breaker was triggered, first at 2:36 p.m., and then at 3:30 p.m. The middle and right panels plot the CSI300 index on January 4 and January 7 of 2016. Trading hours for the Chinese stock market are 9:30-11:30 and 13:00-15:00. Level 1 (2) circuit breaker is triggered after a 5% (7%) drop in price from the previous day's close. The blue circles on the left (right) vertical axes mark the price on the previous day's close (following day's open).

stabilized the next day. This event led to the redesign of the circuit breaker rules, moving from point drops of DJIA to percentage drops of S&P 500, with a considerably wider bandwidth.[4]

After the Chinese stock market experienced extreme price declines in 2015, a market-wide circuit breaker was introduced in January 2016, with a 15-minute trading halt when the CSI 300 Index falls by 5% (Level 1) from previous day's close, and market closure after a 7% decline (Level 2).[5] On January 4, 2016, the first trading day after the circuit breaker was put in place, both thresholds were reached (Figure 1, middle panel), and it took only 7 minutes from the re-opening of the markets following the 15-minute halt

[4]In its current form, the market-wide circuit breaker can be triggered at three thresholds: 7% (Level 1), 13% (Level 2), both of which will halt market-wide trading for 15 minutes when the decline occurs between 9:30 a.m. and 3:25 p.m. Eastern time, and 20% (Level 3), which halts market-wide trading for the remainder of the trading day; these triggers are based on the prior day's closing price of the S&P 500 Index.

[5]The CSI 300 index is a market-cap weighted index of 300 major stocks listed on the Shanghai Stock Exchange and the Shenzhen Stock Exchange, compiled by the China Securities Index Company, Ltd.

for the index to reach the 7% threshold. Three days later, on January 7, both circuit breakers were triggered again (Figure 1, right panel), and the entire trading session lasted just 30 minutes. On the same day, the circuit breaker was suspended indefinitely.

These events have revived debates about circuit breakers. What are the concrete goals for introducing circuit breakers? How may they impact the market? How to assess their success or failure? How may their effectiveness depend on the specific markets, the actual design, and specific market conditions?

In this paper, we develop an intertemporal equilibrium model to capture investors' most fundamental trading needs, namely to share risk. We then examine how the introduction of a downside circuit breaker affects investors' trading behavior and the equilibrium price dynamics. In addition to welfare loss by reduced risk sharing, we show that a circuit breaker also lowers price levels, increases conditional and realized volatility, and increases the likelihood of hitting the triggering point. These consequences are in contrast to the often mentioned goals of circuit breakers. Our model not only demonstrates the potential cost of circuit breakers, but also provides a basic setting to further incorporate market imperfections to fully examine their costs and benefits.

In our model, two (classes of) investors have log preferences over terminal wealth and have heterogeneous beliefs about the dividend growth rate. Without circuit breaker, the stock price is a weighted average of the prices under the two agents' beliefs, with the weights being their respective shares of total wealth.

However, the presence of circuit breakers makes the equilibrium stock price disproportionately reflect the beliefs of the relatively pessimistic investor. To understand this result, first consider the scenario when the stock price has just reached the circuit breaker threshold. Immediate market closure is an extreme form of illiquidity, which forces the relatively optimistic investor to refrain from taking on any leverage due to the inability to rebalance his portfolio and the risk of default it entails. As a result, the pessimistic investor becomes the marginal investors, and the equilibrium stock price has to entirely reflect his beliefs, regardless of his wealth share.

The threat of market closure also affects trading and prices before the circuit breaker is triggered. Compared to the case without circuit breaker, the relatively optimistic investor will preemptively reduce his leverage as the price approaches the circuit breaker limit. For a downside circuit breaker, the price-dividend ratios are driven lower throughout the trading interval. Thus, a downside circuit breaker tends to drive down the overall asset price levels.

In addition, in the presence of a downside circuit breaker, the conditional volatilities of stock returns can become significantly higher. These effects are stronger when the price is closer to the circuit breaker threshold, when it is earlier during a trading session. Surprisingly, the volatility amplification effect of downside circuit breakers is stronger when the initial wealth share for the irrational investor (who tends to be pessimistic at the triggering point) is smaller, because the gap between the wealth-weighted belief of the representative investor and the belief of the pessimist is larger in such cases.

Our model shows that circuit breakers have multifaceted effects on price volatility. On the one hand, almost mechanically, a (tighter) downside circuit breaker limit can lower the median daily price range (measured by daily high minus low prices) and reduce the probabilities of very large daily price ranges. Such effects could be beneficial, for example, in reducing inefficient liquidations due to intra-day mark-to-market. On the other hand, a (tighter) downside circuit breaker will tend to raise the probabilities of intermediate price ranges, and can significantly increase the median of daily realized volatilities as well as the probabilities of very large conditional and realized volatilities. These effects could exacerbate market instability in the presence of imperfections.

Furthermore, our model demonstrates a "magnet effect." The very presence of downside circuit breakers makes it more likely for the stock price to reach the threshold in a given amount of time than when there are no circuit breakers (the opposite is true for upside circuit breakers). The difference between the probabilities is negligible when the stock price is sufficiently far away from the threshold, but it generally gets bigger as the stock price gets closer to the threshold. Eventually, when the price is sufficiently close to the threshold, the gap converges to zero as both probabilities converge to one.

This "magnet effect" is important for the design of circuit breakers. It suggests that using the historical data from a period when circuit breakers were not implemented can lead one to severely underestimate the likelihood of future circuit breaker triggers, which might result in picking a downside circuit breaker limit that is excessively tight.

Prior theoretical work on circuit breakers focuses on their role in reducing excess volatility and restore orderly trading by improving the availability of information and raising confidence among investors. For example, Greenwald and Stein (1991) argue that, in the presence of informational frictions, trading halts can help make more information available to market participants and in turn improve the efficiency of allocations. On the other hand, Subrahmanyam (1994) argue that circuit breakers can increase price volatility by causing investors with exogenous trading demands to advance their trades to earlier periods with lower liquidity supply. Furthermore, building on the insights of Diamond and Dybvig (1983), Bernardo and Welch (2004) show that, when facing the threat of future liquidity shocks, coordination failures can lead to runs and high volatility in the financial market. Such mechanisms could also increase price volatility in the presence of circuit breakers. By building a model to capture investors' first-order trading needs, our work complements these studies in two important dimensions. First, it captures the cost of circuit breakers, in welfare, price level and volatility. Second, it provides a basis to further include different forms of market imperfections such asymmetric information, strategic behavior, failure of coordination, which are needed to justify and quantify the benefits of circuit breakers.

In this spirit, this paper is closely related to Hong and Wang (2000), who study the effects of periodic market closures in the presence of asymmetric information. The liquidity effect caused by market closures as we see here is qualitatively similar to what they find. By modeling the stochastic nature of a circuit breaker, we are able to fully capture its impact on market dynamics, such as volatility and conditional distributions.

While our model focuses on circuit breakers, our main result about the impact of disappearing liquidity on trading and price dynamics is more broadly applicable. Besides market-wide trading halts, other types of market interruptions such as price limits, short-sale ban, trading frequency restrictions (e.g., penalties for HFT), and various forms of liquidity shocks can all have similar effects on the willingness of some investors to take on risky positions, which results in depressed prices and amplified volatility. In fact, the set up we have developed here can be extended to examine these interruptions.

In summary, we provide a new competitive benchmark to demonstrate the potential costs of circuit breakers, including welfare, price level and volatility. Such a benchmark is valuable for several reasons. First, information asymmetry is arguably less important for deep markets, such as the aggregate stock market, than for shallow markets, such as markets for individual securities. Thus, the results we obtain here should be more definitive for market-wide circuit breakers. Second, we show that with competitive investors and complete markets, the threat of (future) market shutdowns can have rich implications such including volatility amplification and self-predatory trading, which will remain present in models involving information asymmetry and strategic behavior. Third, our model sheds light on the behavior of "noise traders" traders in models of information asymmetry, where these investors trade for liquidity reasons but their demands are treated as exogenous. In fact, our results suggest that the behavior of these liquidity traders can be significantly affected by circuit breakers.

The rest of the paper is organized as follows. Section 2 describes the basic model for our analysis. Section 3 provides the solution to the model. In Section 4, we examine the impact of a downside circuit breaker on investor behavior and equilibrium prices. Section 5 discusses the robustness of our results with respect to some of our modelling choices such as continuous-time trading and no default. In Section 6, we consider several extensions of the basic model to different types of trading halts. Section 7 concludes. All proofs are given in the appendix.

2 The Model

We consider a continuous-time endowment economy over the finite time interval $[0, T]$. Uncertainty is described by a one-dimensional standard Brownian motion Z, defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where $\{\mathcal{F}_t\}$ is the augmented filtration generated by Z.

There is a single share of an aggregate stock, which pays a terminal dividend of D_T

at time T. The process for D is exogenous and publicly observable, given by:

$$dD_t = \mu D_t dt + \sigma D_t dZ_t, \quad D_0 = 1, \tag{1}$$

where μ and $\sigma > 0$ are the expected growth rate and volatility of D_t.[6] Besides the stock, there is also a riskless bond with total net supply $\Delta \geq 0$. Each unit of the bond yields a terminal pays off of one at time T.

There are two competitive agents A and B, who are initially endowed with ω and $1-\omega$ shares of the aggregate stock and $\omega\Delta$ and $(1-\omega)\Delta$ units of the riskless bond, respectively, with $0 \leq \omega \leq 1$ determining the initial wealth distribution between the agents. Both agents have logarithmic preferences over their terminal wealth at time T:

$$u_i(W_T^i) = \ln(W_T^i), \quad i = \{A, B\}. \tag{2}$$

There is no intermediate consumption.

The two agents have heterogeneous beliefs about the terminal dividend, and they "agree to disagree" (i.e., they do not learn from each other or from prices). Agent A has the objective beliefs in the sense that his probability measure is consistent with $\mathbb{P}$ (in particular, $\mu^A = \mu$). Agent B's probability measure, denoted by $\mathbb{P}^B$, is different from but equivalent to $\mathbb{P}$.[7] In particular, he believes that the growth rate at time t is:

$$\mu_t^B = \mu + \delta_t, \tag{3}$$

where the difference in beliefs δ_t follows an Ornstein-Uhlenbeck process:

$$d\delta_t = -\kappa(\delta_t - \bar{\delta})dt + \nu dZ_t, \tag{4}$$

with $\kappa \geq 0$ and $\nu \geq 0$. Equation (4) describes the dynamics of the gap in beliefs from the

[6] For brevity, throughout the paper we will refer to D_t as "dividend" and S_t/D_t as the "price-dividend ratio," even though dividend will only be realized at time T.

[7] More precisely, $\mathbb{P}$ and $\mathbb{P}^B$ are equivalent when restricted to any σ-field $\mathcal{F}_T = \sigma(\{D_t\}_{0 \leq t \leq T})$. Two probability measures are equivalent if they agree on zero probability events. Agents beliefs should be equivalent to prevent seemingly arbitrage opportunities under any agents' beliefs.

perspective of agent A (the physical probability measure). Notice that δ_t is driven by the same Brownian motion as the aggregate dividend. With $\nu > 0$, agent B becomes more optimistic (pessimistic) following positive (negative) shocks to the aggregate dividend, and the impact of these shocks on his belief decays exponentially at the rate κ. Thus, the parameter ν controls how sensitive B's conditional belief is to realized dividend shocks, while κ determines the relative importance of shocks from recent past vs. distant past. The average long-run disagreement between the two agents is $\bar{\delta}$. In the special case with $\nu = 0$ and $\delta_0 = \bar{\delta}$, the disagreement between the two agents remains constant over time. In another special case where $\kappa = 0$, δ_t follows a random walk.

It is worth pointing out that allowing heterogeneous beliefs is a simple way to introduce heterogeneity among agents, which motivates trading. The heterogeneity in beliefs can easily be interpreted as heterogeneity in utility, which can be state dependent. For example, time-varying beliefs could represent behavioral biases ("representativeness") or a form of path-dependent utility that makes agent B more (less) risk averse following negative (positive) shocks to fundamentals. Alternatively, we could introduce heterogeneous endowment shocks to generate trading (see, e.g., Wang (1995)). In all these cases, trading allows agents to share risk.

Let the Radon-Nikodym derivative of the probability measure $\mathbb{P}^B$ with respect to $\mathbb{P}$ be η. Then from Girsanov's theorem, we get

$$\eta_t = \exp\left(\frac{1}{\sigma}\int_0^t \delta_s dZ_s - \frac{1}{2\sigma^2}\int_0^t \delta_s^2 ds\right). \tag{5}$$

Intuitively, since agent B will be more optimistic than A when $\delta_t > 0$, those paths with high realized values for $\int_0^t \delta_s dZ_s$ will be assigned higher probabilities under $\mathbb{P}^B$ than under $\mathbb{P}$.

Because there is no intermediate consumption, we use the riskless bond as the numeraire. Thus, the price of the bond is always 1.

Circuit Breaker. To capture the essence of a circuit breaker rule, we assume that the stock market will be closed whenever the price of the stock S_t falls below a threshold

$(1-\alpha)S_0$, where S_0 is the endogenous initial price of the stock, and $\alpha \in [0,1]$ is a constant parameter determining the bandwidth of downside price fluctuations during the interval $[0,T]$. Later in Section 6, we extend the model to allow for market closures for both downside and upside price movements, which represent price limit rules. The closing price for the stock is determined such that both the stock market and bond market are cleared when the circuit breaker is triggered. After that, the stock market will remain closed until time T. The bond market remains open throughout the interval $[0,T]$.

In practice, the circuit breaker threshold is often based on the closing price from the previous trading session instead of the opening price of the current trading session. For example, in the U.S., a cross-market trading halt can be triggered at three circuit breaker thresholds (7%, 13%, and 20%) based on the prior day's closing price of the S&P 500 Index. However, the distinction between today's opening price and the prior day's closing price is not crucial for our model. The circuit breaker not only depends on but also endogenously affects the initial stock price, just like it does for prior day's closing price in practice.[8]

Finally, we impose usual restrictions on trading strategies to rule out arbitrage.

3 The Equilibrium

3.1 Benchmark Case: No Circuit Breaker

In this section, we solve for the equilibrium when there is no circuit breaker. To distinguish the notations from the case with circuit breakers, we use the symbol "$\widehat{\ }$" to denote variables in the case without circuit breakers.

In the absence of circuit breakers, markets are dynamically complete. The equilibrium allocation in this case can be characterized as the solution to the following planner's

[8]Other realistic features of the circuit breaker in practice is to close the market for m minutes and reopen (Level 1 and 2), or close the market until the end of the day (Level 3). In our model, we can think of T as one day. The fact that the price of the stock reverts back to the fundamental value X_T at T resembles the rationale of CB to "restore order" in the market.

problem:

$$\max_{\widehat{W}_T^A, \widehat{W}_T^B} \mathbb{E}_0 \left[\lambda \ln \left(\widehat{W}_T^A \right) + (1-\lambda)\eta_T \ln \left(\widehat{W}_T^B \right) \right], \tag{6}$$

subject to the resource constraint

$$\widehat{W}_T^A + \widehat{W}_T^B = D_T + \Delta. \tag{7}$$

From the first-order conditions and the budget constraints, we then get $\lambda = \omega$, and

$$\widehat{W}_T^A = \frac{\omega}{\omega + (1-\omega)\eta_T}(D_T + \Delta), \tag{8}$$

$$\widehat{W}_T^B = \frac{(1-\omega)\eta_T}{\omega + (1-\omega)\eta_T}(D_T + \Delta). \tag{9}$$

As it follows from the equations above agent B will be allocated a bigger share of the aggregate dividend when realized value of the Radon-Nikodym derivative η_T is higher, i.e., under those paths that agent B considers to be more likely.

The state price density under agent A's beliefs, which is also the objective probability measure $\mathbb{P}$, is

$$\widehat{\pi}_t^A = \mathbb{E}_t \left[\xi u'(\widehat{W}_T^A) \right] = \mathbb{E}_t \left[\xi (\widehat{W}_T^A)^{-1} \right], \quad 0 \le t \le T \tag{10}$$

for some constant ξ. Then, from the budget constraint for agent A we see that the planner's weights are equal to the shares of endowment, $\lambda = \theta$. Using the state price density, one can then derive the price of the stock and individual investors' portfolio holdings.

In the limiting case with bond supply $\Delta \to 0$, the complete markets equilibrium can be characterized in closed form. We focus on this limiting case in the rest of the section. First, the following proposition summarizes the pricing results.

Proposition 1. *When there are no circuit breakers, the price of the stock in the limiting case with bond supply $\Delta \to 0$ is:*

$$\widehat{S}_t = \frac{\omega + (1-\omega)\eta_t}{\omega + (1-\omega)\eta_t e^{a(t,T)+b(t,T)\delta_t}} D_t e^{\left(\mu - \sigma^2\right)(T-t)}, \tag{11}$$

where

$$a(t,T) = \left[\frac{\kappa\bar{\delta} - \sigma\nu}{\frac{\nu}{\sigma} - \kappa} + \frac{\nu^2}{2\left(\frac{\nu}{\sigma} - \kappa\right)^2}\right](T-t) - \frac{\nu^2}{4\left(\frac{\nu}{\sigma} - \kappa\right)^3}\left[1 - e^{2\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}\right]$$
$$+ \left[\frac{\kappa\bar{\delta} - \sigma\nu}{\left(\frac{\nu}{\sigma} - \kappa\right)^2} + \frac{\nu^2}{\left(\frac{\nu}{\sigma} - \kappa\right)^3}\right]\left[1 - e^{\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}\right], \tag{12}$$
$$b(t,T) = \frac{1 - e^{\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}}{\frac{\nu}{\sigma} - \kappa}. \tag{13}$$

From Equation (11), we can derive the conditional volatility of the stock $\widehat{\sigma}_{S,t}$ in closed form, which is available in the appendix.

Next, we turn to the wealth distribution and portfolio holdings of individual agents. At time $t \leq T$, the shares of total wealth of the two agents are:

$$\widehat{\omega}_t^A = \frac{\omega}{\omega + (1-\omega)\eta_t}, \quad \widehat{\omega}_t^B = 1 - \widehat{\omega}_t^A. \tag{14}$$

The number of shares of stock $\widehat{\theta}_t^A$ and units of riskless bonds $\widehat{\phi}_t^A$ held by agent A are:

$$\widehat{\theta}_t^A = \frac{\omega}{\omega + (1-\omega)\eta_t} - \frac{\omega(1-\omega)\eta_t}{\left[\omega + (1-\omega)\eta_t\right]^2}\frac{\delta_t}{\sigma\widehat{\sigma}_{S,t}} = \widehat{\omega}_t^A\left(1 - \widehat{\omega}_t^B\frac{\delta_t}{\sigma\widehat{\sigma}_{S,t}}\right), \tag{15}$$
$$\widehat{\phi}_t^A = \widehat{\omega}_t^A\widehat{\omega}_t^B\frac{\delta_t}{\sigma\widehat{\sigma}_{S,t}}\widehat{S}_t, \tag{16}$$

and the corresponding values for agent B are $\theta_t^B = 1 - \theta_t^A$ and $\phi_t^B = -\phi_t^A$.

As Equation (15) shows, there are several forces affecting the portfolio positions. First, all else equal, agent A owns fewer shares of the stock when B has more optimistic beliefs (larger δ_t). This effect becomes weaker when the volatility of stock return $\widehat{\sigma}_{S,t}$ is high. Second, changes in the wealth distribution (as indicated by (14)) also affect the portfolio holdings, as the richer agent will tend to hold more shares of the stock.

We can gain more intuition on the stock price by rewriting Equation (11) as follows:

$$\widehat{S}_t = \frac{1}{\frac{\omega}{\omega+(1-\omega)\eta_t}\mathbb{E}_t\left[D_T^{-1}\right] + \frac{(1-\omega)\eta_t}{\omega+(1-\omega)\eta_t}\mathbb{E}_t^B\left[D_T^{-1}\right]} = \left(\frac{\widehat{\omega}_t^A}{\widehat{S}_t^A} + \frac{\widehat{\omega}_t^B}{\widehat{S}_t^B}\right)^{-1}, \tag{17}$$

which states that the stock price is a weighted harmonic average of the prices of the stock in two single-agent economies with agent A and B being the representative agent, $\widehat{S}_t^A$ and $\widehat{S}_t^B$, where

$$\widehat{S}_t^A = e^{\left(\mu-\sigma^2\right)(T-t)} D_t, \tag{18}$$

$$\widehat{S}_t^B = e^{\left(\mu-\sigma^2\right)(T-t)-a(t,T)-b(t,T)\delta_t} D_t, \tag{19}$$

and the weights $(\widehat{\omega}_t^A, \widehat{\omega}_t^B)$ are the two agents' shares of total wealth. For example, controlling for the wealth distribution, the equilibrium stock price is higher when agent B has more optimistic beliefs (larger δ_t).

One special case of the above result is when the amount of disagreement between the two agents is the zero, i.e., $\delta_t = 0$ for all $t \in [0, T]$. The stock price then becomes:

$$\widehat{S}_t = \widehat{S}_t^A = \frac{1}{\mathbb{E}_t[D_T^{-1}]} = e^{\left(\mu-\sigma^2\right)(T-t)} D_t, \tag{20}$$

which is a version of the Gordon growth formula, with σ^2 being the risk premium for the stock. The instantaneous volatility of stock returns becomes the same as the volatility of dividend growth, $\widehat{\sigma}_{S,t} = \sigma$. The shares of the stock held by the two agents will remain constant and be equal to the their endowments, $\widehat{\theta}_t^A = \omega$, $\widehat{\theta}_t^B = 1 - \omega$.

Another special case is when the amount of disagreement is constant over time ($\delta_t = \delta$ for all t). The results for this case are obtained by setting $\nu = 0$ and $\delta_0 = \bar{\delta} = \delta$ in Proposition 1. In particular, Equation (11) simplifies to:

$$\widehat{S}_t = \frac{\omega + (1-\omega)\eta_t}{\omega + (1-\omega)\eta_t e^{-\delta(T-t)}} e^{\left(\mu-\sigma^2\right)(T-t)} D_t. \tag{21}$$

3.2 Circuit Breaker

We start this section by introducing some notation. By θ_t^i, ϕ_t^i, and W_t^i we denote stock holdings, bond holdings, and wealth of agent i at time t, respectively, in the market with a circuit breaker. Let τ denote the time when the circuit breaker is triggered. It follows from the definition of the circuit breaker and the continuity of stock prices that τ satisfies

$$\tau = \inf\{t \geq 0 : S_t = (1-\alpha)S_0\}. \tag{22}$$

We use the expression $\tau \wedge T$ to denote $\min\{\tau, T\}$. Next, we define the equilibrium with a circuit breaker.

Definition 1. *The equilibrium with circuit breaker is defined by an $\mathcal{F}_t$-stopping time τ, trading strategies $\{\theta_t^i, \phi_t^i\}$ $(i = A, B)$, and a continuous stock price process S defined on the interval $[0, \tau \wedge T]$ such that:*

1. *Taking stock price process S as given, the trading strategies maximize the two agents' expected utilities under their respective beliefs subject to the budget constraints.*
2. *For any $t \in [0, T]$, both the stock and bond markets clear,*

$$\theta_t^A + \theta_t^B = 1, \quad \phi_t^A + \phi_t^B = \Delta. \tag{23}$$

3. *The stopping time τ is consistent with the circuit breaker rule in (22).*

One crucial feature that facilitates solving the model is that markets remain dynamically complete until the circuit breaker is triggered. Hence, we solve for an equilibrium with the following three steps. First, consider an economy in which trading stops when the stock price reaches any given triggering price $\underline{S} \geq 0$. By examining the equilibrium conditions upon market closure, we can characterize the $\mathcal{F}_t$-stopping time τ that is consistent with $S_\tau = \underline{S}$. Next, we can solve for the optimal allocation at $\tau \wedge T$ through the planner's problem as a function of $\underline{S}$, as well as the stock price prior to $\tau \wedge T$, again as a function of $\underline{S}$. Finally, the equilibrium is the fixed point whereby the triggering price $\underline{S}$ is consistent with the initial price, $\underline{S} = (1-\alpha)S_0$. We describe these steps in detail below.

Suppose the circuit breaker is triggered before the end of the trading session, i.e., $\tau < T$. We start by deriving the agents' indirect utility functions at the time of market closure. Agent i has wealth W_τ^i at time τ. Since the two agents behave competitively, they take the stock price S_τ as given and choose the shares of stock θ^i_τ and bonds ϕ^i_τ to maximize their expected utility over terminal wealth, subject to the budget constraint:

$$V^i(W^i_\tau, \tau) = \max_{\theta^i_\tau, \phi^i_\tau} \mathbb{E}^i_\tau \left[\ln(\theta^i_\tau D_T + \phi^i_\tau) \right], \tag{24}$$

$$s.t. \quad \theta^i_\tau S_\tau + \phi^i_\tau = W^i_\tau, \tag{25}$$

where $V^i(W^i_\tau, \tau)$ is the indirect utility function for agent i at time $\tau < T$.

The market clearing conditions at time τ are:

$$\theta^A_\tau + \theta^B_\tau = 1, \quad \phi^A_\tau + \phi^B_\tau = \Delta. \tag{26}$$

For any $\tau < T$, the Inada condition implies that terminal wealth for both agents needs to stay non-negative, which implies $\theta^i_\tau \geq 0$, $\phi^i_\tau \geq 0$. That is, neither agent will take short or levered positions in the stock. This is a direct result of the inability to rebalance one's portfolio after market closure, which is an extreme version of illiquidity.

Solving the problem (24) – (26) gives us the indirect utility functions $V^i(W^i_\tau, \tau)$. It also gives us the stock price at the time of market closure, S_τ, as a function of the dividend D_τ, the gap in believes δ_τ, and the wealth distribution at time τ (which is determined by the Radon-Nikodym derivative η_τ). Thus, the condition $S_\tau = \underline{S}$ translates into a constraint on D_τ, δ_τ, and η_τ, which in turn characterizes the stopping time τ as a function of exogenous state variables. As we will see later, in the limiting case with bond supply $\Delta \to 0$, the stopping rule satisfying this restriction can be expressed in closed form. When $\Delta > 0$, the solution can be obtained numerically.

Next, the indirect utility for agent i at $\tau \wedge T$ is given by:

$$V^i(W^i_{\tau\wedge T}, \tau \wedge T) = \begin{cases} \ln(W^i_T), & \text{if} \quad \tau \geq T \\ V^i(W^i_\tau, \tau), & \text{if} \quad \tau < T \end{cases} \tag{27}$$

These indirect utility functions make it convenient to solve for the equilibrium wealth allocations in the economy at time $\tau \wedge T$ through the following planner problem:

$$\max_{W^A_{\tau\wedge T}, W^B_{\tau\wedge T}} \mathbb{E}_0 \left[\lambda V^A(W^A_{\tau\wedge T}, \tau \wedge T) + (1-\lambda)\eta_{\tau\wedge T} V^B(W^B_{\tau\wedge T}, \tau \wedge T)\right], \tag{28}$$

subject to the resource constraint:

$$W^A_{\tau\wedge T} + W^B_{\tau\wedge T} = S_{\tau\wedge T} + \Delta, \tag{29}$$

where

$$S_{\tau\wedge T} = \begin{cases} D_T, & \text{if} \quad \tau \geq T \\ \underline{S}, & \text{if} \quad \tau < T \end{cases} \tag{30}$$

Taking the equilibrium allocation $W^A_{\tau\wedge T}$ from the planner's problem, the state price density for agent A at time $\tau \wedge T$ can be expressed as his marginal utility of wealth times a constant ξ,

$$\pi^A_{\tau\wedge T} = \xi \frac{\partial V^A(W, \tau \wedge T)}{\partial W}\Big|_{W=W^A_{\tau\wedge T}}. \tag{31}$$

The price of the stock at any time $t \leq \tau \wedge T$ is then given by:

$$S_t = \mathbb{E}_t \left[\frac{\pi^A_{\tau\wedge T}}{\pi^A_t} S_{\tau\wedge T}\right], \tag{32}$$

where like in Equation (10),

$$\pi^A_t = \mathbb{E}_t \left[\pi^A_{\tau\wedge T}\right]. \tag{33}$$

The expectations above are straightforward to evaluate, at least numerically. Having obtained the solution for S_t as a function of $\underline{S}$, we can finally solve for the equilibrium triggering price $\underline{S}$ through the following fixed point problem,

$$\underline{S} = (1-\alpha) S_0. \tag{34}$$

Proposition 2. *There exists a solution to the fixed-point problem in* (34) *for any* $\alpha \in [0, 1]$.

To see why Proposition 2 holds, consider S_0 as a function of $\underline{S}$, $S_0 = f(\underline{S})$. First notice that when $\underline{S} = 0$, there is essentially no circuit breaker, and $f(0)$ will be the same as the initial stock price in the complete markets case. Next, there exists $s^* > 0$ such that $s^* = f(s^*)$, which is the initial price when the market closes immediately after opening. The fact that f is continuous ensures that there exists at least one crossing between the function $f(s)$ and $s/(1-\alpha)$, which will be a solution for (34).

Below we will show how these steps can be neatly solved in the special case when riskless bonds are in zero net supply.

Because neither agent will take levered or short positions during market closure, there cannot be any lending or borrowing in that period. Thus, in the limiting case with net bond supply $\Delta \to 0$, all the wealth of the two agents will be invested in the stock upon market closure, and consequently the leverage constraint will always bind for the relatively optimistic investor in the presence of heterogeneous beliefs. The result is that the relatively pessimistic investor becomes the marginal investor, as summarized in the following proposition.

Proposition 3. *Suppose the stock market closes at time $\tau < T$. In the limiting case with bond supply $\Delta \to 0$, both agents will hold all of their wealth in the stock, $\theta_\tau^i = \frac{W_\tau^i}{S_\tau}$, and hold no bonds, $\phi_\tau^i = 0$. The market clearing price is:*

$$S_\tau = \min\{\widehat{S}_\tau^A, \widehat{S}_\tau^B\} = \begin{cases} e^{\left(\mu-\sigma^2\right)(T-\tau)} D_\tau, & \text{if} \quad \delta_\tau > \underline{\delta}(\tau) \\ e^{\left(\mu-\sigma^2\right)(T-\tau)-a(\tau,T)-b(\tau,T)\delta_\tau} D_\tau, & \text{if} \quad \delta_\tau \leq \underline{\delta}(\tau) \end{cases} \tag{35}$$

where $\widehat{S}_\tau^i$ denotes the stock price in a single-agent economy populated by agent i, as given in (18)-(19),

$$\underline{\delta}(t) = -\frac{a(t,T)}{b(t,T)}, \tag{36}$$

and $a(t,T)$, $b(t,T)$ are given in Proposition 1.

Notice that the market clearing price S_τ only depends on the belief of the relatively pessimistic agent. This result is qualitatively different from the complete markets case, where the stock price is a wealth-weighted average of the prices under the two agents' beliefs. It is a crucial result: the lower stock valuation upon market closure affects both the stock price level and dynamics before market closure, which we analyze in Section 4. Notice that having the lower expectation of the growth rate at the current instant is not sufficient to make the agent marginal. One also needs to take into account the agents' future beliefs and the risk premium associated with future fluctuations in the beliefs, which are summarized by $\underline{\delta}(t)$.[9]

Equation (35) implies that we can characterize the stopping time τ using a stochastic threshold for dividend D_t, as summarized below.

Lemma 1. *Take the triggering price $\underline{S}$ as given. Define a stopping time*

$$\tau = \inf\{t \geq 0 : D_t = \underline{D}(t, \delta_t)\}, \tag{37}$$

where

$$\underline{D}(t, \delta_t) = \begin{cases} \underline{S}e^{-\left(\mu-\sigma^2\right)(T-t)}, & \text{if} \quad \delta_t > \underline{\delta}(t) \\ \underline{S}e^{-\left(\mu-\sigma^2\right)(T-t)+a(t,T)+b(t,T)\delta_t}, & \text{if} \quad \delta_t \leq \underline{\delta}(t) \end{cases} \tag{38}$$

Then, in the limiting case with bond supply $\Delta \to 0$, the circuit breaker is triggered at time τ whenever $\tau < T$.

Having characterized the equilibrium at time $\tau < T$, we plug the equilibrium portfolio holdings into (24) to derive the indirect utility of the two agents at τ:

$$V^i(W^i_\tau, \tau) = \mathbb{E}^i_\tau\left[\ln\left(\frac{W^i_\tau}{S_\tau}D_T\right)\right] = \ln(W^i_\tau) - \ln(S_\tau) + \mathbb{E}^i_\tau[\ln(D_T)]. \tag{39}$$

The indirect utility for agent i at $\tau \wedge T$ is then given by:

$$V^i(W^i_{\tau\wedge T}, \tau \wedge T) = \begin{cases} \ln(W^i_T), & \text{if} \quad \tau \geq T \\ \ln(W^i_\tau) - \ln(S_\tau) + \mathbb{E}^i_\tau[\ln(D_T)], & \text{if} \quad \tau < T \end{cases} \tag{40}$$

[9] Technically, there is a difference between the limiting case with $\Delta \to 0$ and the case with $\Delta = 0$. When $\Delta = 0$, any price equal or below S_τ in (35) will clear the market. At such prices, both agents would prefer to invest more than 100% of their wealth in the stock, but both will face binding leverage constraints, which is why the stock market clears at these prices. However, these alternative equilibria are ruled out by considering a sequence of economies with bond supply $\Delta \to 0$. In each of these economies where $\Delta > 0$, the relatively pessimistic agent needs to hold the bond in equilibrium, which means his leverage constraint cannot not be binding.

Substituting these indirect utility functions into the planner's problem (28) and taking the first order condition, we get the wealth of agent A at time $\tau \wedge T$:

$$W^A_{\tau\wedge T} = \frac{\omega S_{\tau\wedge T}}{\omega + (1-\omega)\,\eta_{\tau\wedge T}}, \tag{41}$$

where $S_{\tau\wedge T}$ is given in (30). Then, we obtain the state price density for agent A and the price of the stock at time $t \leq \tau \wedge T$ as in (31) and (32), respectively. In particular,

$$S_t = \left(\omega^A_t \mathbb{E}_t\left[S^{-1}_{\tau\wedge T}\right] + \omega^B_t \mathbb{E}^B_t\left[S^{-1}_{\tau\wedge T}\right]\right)^{-1}. \tag{42}$$

Here ω^i_t is the share of total wealth owned by agent i, which, in the limiting case with $\Delta \to 0$, is identical to $\widehat{\omega}^i_t$ in (14) before market closure. Equation (42) is reminiscent of its complete markets counterpart (17). Unlike in the case of complete markets, the expectations in (42) are no longer the inverse of the stock prices from the respective representative agent economies.

From the stock price, we can then compute the conditional mean $\mu_{S,t}$ and volatility $\sigma_{S,t}$ of stock returns, which are given by

$$dS_t = \mu_{S,t}S_t dt + \sigma_{S,t}S_t dZ_t. \tag{43}$$

In Appendix A.3, we provide the closed-form solution for S_t in the special case with constant disagreements ($\delta_t \equiv \delta$).

Finally, by evaluating S_t at time $t = 0$, we can solve for $\underline{S}$ from the fixed point problem (34). Beyond the existence result of Proposition 2, one can further show that the fixed point is unique when the riskless bond is in zero net supply.[10]

The case of positive bond supply. When the riskless bond is in positive net supply, there are four possible scenarios upon market closure: the relatively optimistic agent faces binding leverage constraint, while the relatively pessimistic agent is either uncon-

[10]The uniqueness is due to the fact that S_0 will be monotonically decreasing in the triggering price $\underline{S}$ in the limiting case when $\Delta \to 0$, which is not necessarily true when $\Delta > 0$.

strained (i) or faces binding short-sale constraint (ii); the relatively optimistic agent is unconstrained, while the relatively pessimistic agent is either unconstrained (iii) or faces binding short-sale constraint (iv). In contrast, only Scenario (i) is possible in the case where the riskless bond is in zero net supply. The three new scenarios originate from the fact that when $\Delta > 0$ the two agents can hold different portfolios without borrowing and lending; furthermore, when the relatively optimistic agent is sufficiently wealthy, he could potentially hold the entire stock market without having to take on any leverage.

In particular, Scenario (iv) is the opposite of Scenario (i) in that the relatively optimistic agent, instead of the pessimistic one, becomes the marginal investor. As a result, the price level can become higher and volatility lower in the economy with a circuit breaker. Under Scenarios (ii) and (iii), the equilibrium stock price upon market closure is somewhere in between the two agents' valuations.

In Section 4.3, we examine the conditions (wealth distribution, size of bond supply, and amount of disagreement upon market closure) that determine which of the scenarios occur in equilibrium. As we show later, which of the scenarios is realized has important implications for the equilibrium price process.

Circuit breaker and wealth distribution. We conclude this section by examining the impact of circuit breakers on the wealth distribution. As explained earlier, the wealth shares of the two agents before market closure (at time $t \leq \tau \wedge T$) will be the same as in the economy without circuit breakers, and take the form in (14) when the riskless bonds are in zero net supply.

However, the wealth shares at the end of the trading day (time T) will be affected by the presence of the circuit breaker. This is because if the circuit breaker is triggered at $\tau < T$, the wealth distribution after τ will remain fixed due to the absence of trading. Since irrational traders on average lose money over time, market closure at $\tau < T$ will raise their average wealth share at time T. This "mean effect" implies that circuit breakers will help "protecting" the irrational investors in this model. How strong this effect is depends on the amount of disagreement and the distribution of τ. In addition, circuit breakers will also make the tail of the wealth share distribution thinner as they put a limit on the amount of wealth that the relatively optimistic investor can lose over time along those paths with low realizations of D_t.

4 Impact of Circuit Breakers on Market Dynamics

We now turn to the quantitative implications of the model. First, in Section 4.1 we examine the special case of the model with zero net supply of riskless bonds and constant disagreement, i.e., $\Delta \to 0$ and $\delta_t \equiv \delta$. This case helps demonstrate the main mechanism through which circuit breakers affect asset prices and trading. Then, in Section 4.2, we examine the full model featuring time-varying disagreements while riskless bonds are still in zero net supply. Finally, we investigate the impact of positive bond supply on the main implications of our model in Section 4.3.

4.1 Constant Disagreement

For calibration, we normalize $T = 1$ to denote one trading day. We set $\mu = 10\%/250 = 0.04\%$ (implying an annual dividend growth rate of 10%), and we assume daily volatility of dividend growth $\sigma = 3\%$. The circuit breaker threshold is set at $\alpha = 5\%$. For the initial wealth distribution, we assume agent A (with rational beliefs) owns 90% of total wealth ($\omega = 0.9$) at $t = 0$. For the amount of disagreement, we set $\delta = -2\%$, which means agent B is relatively pessimistic about dividend growth, and his valuation of the stock at $t = 0$, $\widehat{S}_0^B$, will be 2% lower than that of agent A, $\widehat{S}_0^A$, which is fairly modest.

In Figure 2, we plot the equilibrium price-dividend ratio S_t/D_t (left column), the conditional volatility of returns (middle column), and the stock holding for agent A (right column). The stock holding for agent B can be inferred from that of agent A, as $\theta_t^B = 1 - \theta_t^A$. In each panel, the solid line denotes the solution for the case with circuit breaker, while the dotted line denotes the case without circuit breaker. To examine the time-of-the-day effect, we plot the solutions at two different points in time, $t = 0.25, 0.75$.

Let's start with the price-dividend ratio. As discussed in Section 3.1, the price of the stock in the case without circuit breaker is the weighted (harmonic) average of the

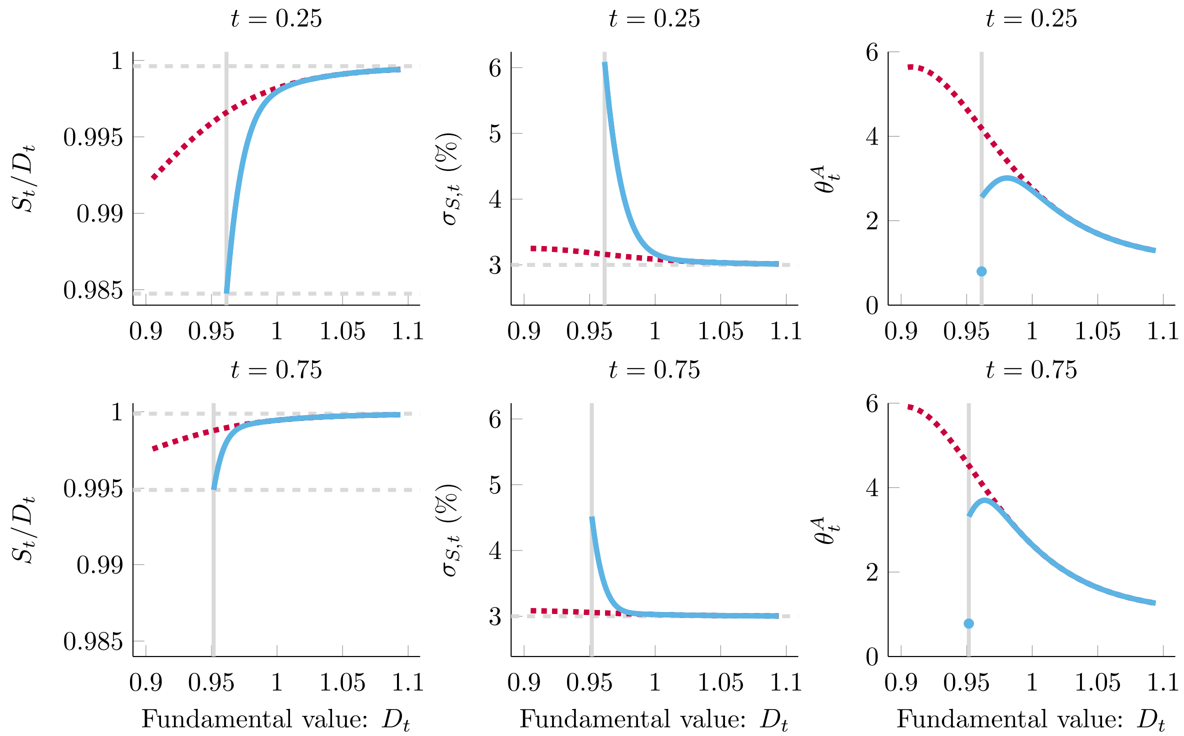

Figure 2: **Price-dividend ratio, conditional return volatility, and agent A's (rational optimist) portfolio holding.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

prices of the stock from the two representative-agent economies populated by agent A and B, respectively, with the weights given by the two agents' shares of total wealth (see equation (17)). Under our calibration, the price-dividend ratio is close to one for any $t \in [0, T]$ under agent A's beliefs ($\widehat{S}_t^A/D_t$), and it is approximately equal to $e^{\delta(T-t)} \leq 1$ under agent B's beliefs ($\widehat{S}_t^B/D_t$). These two values are denoted by the upper and lower horizontal dash lines in the left column of Figure 2.

The price-dividend ratio in the economy without circuit breaker (red dotted line) indeed lies between $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$. Since agent A is relatively more optimistic, he will hold levered position in the stock (see the red dotted line in the middle column), and his share of total wealth will become higher following positive shocks to the dividend. Thus, as dividend value D_t rises (falls), the share of total wealth owned by agent A increases (decreases), which makes the equilibrium price-dividend ratio approach the value $\widehat{S}_t^A/D_t$ ($\widehat{S}_t^B/D_t$).

In the case with circuit breaker, the price-dividend ratio (blue solid line) still lies between the price-dividend ratios from the two representative agent economies, but it is always below the price-dividend ratio without circuit breaker for a given level of dividend. The gap between the two price-dividend ratios is negligible when D_t is sufficiently high, but it widens as D_t approaches the circuit breaker threshold $\underline{D}(t)$.

The reason that stock price declines more rapidly with dividend in the presence of a circuit breaker can be traced to how the stock price is determined upon market closure. As explained in Section 3.2, at the instant when the circuit breaker is triggered, neither agent will be willing to take on levered position in the stock due to the inability to rebalance the portfolio. With bonds in zero net supply, the leverage constraint always binds for the relatively optimistic agent (agent A), and the market clearing stock price has to be such that agent B is willing to hold all of his wealth in the stock, *regardless of his share of total wealth*. Indeed, we see the price-dividend ratio with circuit breaker converging to $\widehat{S}_t^B/D_t$ when D_t approaches $\underline{D}(t)$, instead of the wealth-weighted average of $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$. The lower stock price at the circuit breaker threshold also drives the stock price lower before market closure, with the effect becoming stronger as D_t moves closer to the threshold $\underline{D}(t)$. This explains the accelerated decline in stock price as D_t drops.

The higher sensitivity of the price-dividend ratio to dividend shocks due to the circuit breaker manifests itself in elevated conditional return volatility, as shown in the middle column of Figure 2. Quantitatively, the impact of the circuit breaker on the conditional volatility of stock returns can be quite sizable. Without circuit breaker, the conditional volatility of returns (red dotted lines) peaks at about 3.2%, only slightly higher than the fundamental volatility of $\sigma = 3\%$. This small amount of excess volatility comes from the time variation in the wealth distribution between the two agents. With circuit breaker, the conditional volatility (blue solid lines) becomes substantially higher as D_t approaches $\underline{D}(t)$. For example, when $t = 0.25$, the conditional volatility reaches 6% at the circuit breaker threshold, almost twice as high as the return volatility without circuit breaker.

We can also analyze the impact of the circuit breaker on the equilibrium stock price by connecting it to how the circuit breaker influences the equilibrium portfolio holdings of the two agents. Let us again start with the case without circuit breaker (red dotted lines in right column of Figure 2). The stock holding of agent A, $\widehat{\theta}_t^A$, continues to rise as D_t falls to $\underline{D}(t)$ and beyond. This is the result of two effects: (i) with lower D_t, the stock price is lower, implying higher expected return under agent A's beliefs; (ii) lower D_t also makes agent B (who is shorting the stock) wealthier and thus more capable of lending to agent A, who then takes on a more levered position.

With circuit breaker, while the stock holding θ_t^A takes on similar values as $\widehat{\theta}_t^A$, its counterpart in the case without circuit breaker, for large values of D_t, it becomes visibly lower than $\widehat{\theta}_t^A$ as D_t approaches the circuit breaker threshold, and it eventually starts to decrease as D_t continues to drop. This is because agent A becomes increasingly concerned with the rising return volatility at lower D_t, which eventually dominates the effect of higher expected stock return. Finally, θ_t^A takes a discrete drop to 1 when $D_t = \underline{D}(t)$, as the leverage constraint becomes binding. The scaling-back of stock holding by agent A even before the circuit breaker is triggered can be interpreted as a form of "self-predatory" trading. The stock price in equilibrium has to fall enough such that agent A has no incentive to sell more of his stock holding.

Time-of-the-day effect. Comparing the cases with $t = 0.25$ and $t = 0.75$, we see that the impact of circuit breaker on the equilibrium price-dividend ratio weakens as t approaches T for any given level of D_t. This is because a shorter remaining horizon reduces the potential impact of agent B's pessimistic beliefs on the equilibrium stock price, as reflected in the shrinking gap between $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$ (the two horizontal dash lines) from the top left panel to the bottom left panel. As a result, the impact of circuit breaker on the price-dividend ratio and the conditional return volatility both weakens. For example, at $t = 0.25$, the price-dividend ratio with circuit breaker can be as much as 1.2% lower than the level without circuit breaker. At $t = 0.75$, the gap is at most 0.3%. When $t = 0.75$, the conditional return volatility peaks at 4.5% at the circuit breaker threshold, compared to 6% when $t = 0.25$. Finally, because the price-dividend ratio becomes higher, the circuit breaker threshold $\underline{D}(t)$ becomes lower as t increases. That is, the dividend level needs to drop more to trigger the circuit breaker.

4.2 Time-varying Disagreement

In the previous section, we use the special case of constant disagreement to illustrate the impact of circuit breakers on trading and price dynamics. We now turn to the full model with time-varying disagreement, where the difference in beliefs δ_t follows a random walk. We do so by setting $\kappa = 0$, $\nu = \sigma$, and $\delta_0 = 0$. Thus, there is neither initial nor long-term bias in agent B's belief. For bond supply, we continue with the limiting case with $\Delta \to 0$.

Under this specification, Agent B's beliefs resemble the "representativeness" bias in behavioral economics. As a form of non-Bayesian updating, he extrapolates his belief about future dividend growth from the realized path of dividend.[11] As a result, he becomes overly optimistic following large positive dividend shocks and overly pessimistic following large negative dividend shocks. An alternative interpretation of such beliefs is that they capture in reduced form the behavior of constrained investors, who become effectively more (less) pessimistic or risk averse as the constraint tightens (loosens).

Figure 3 shows the price-dividend ratio (left column), agent A's stock holding (middle column), and his bond holding (right column); Figure 4 shows the conditional return volatility (left column), and the conditional expected returns under the two agents' beliefs (middle and right column). Unlike in the constant disagreement case, dividend value D_t and time of the day t are no longer sufficient to determine the state of the economy in the case of time-varying disagreement. Thus, we plot the average values of the variables above conditional on t and D_t.[12]

Let's start with the price-dividend ratio, shown in the left column of Figure 3. As in the case of constant disagreement, the price-dividend ratio in the equilibrium without circuit breaker (red dotted line) is still a wealth-weighted average of the price-dividend ratios under the two agents' beliefs. Since agent A's belief about the dividend growth rate is constant over time, the price-dividend ratio under his beliefs is constant over different values of D_t (the horizontal grey dash line). However, due to the variation in δ_t

[11] Specifically, $\delta_t = \ln\left(\frac{D_t}{D_0} e^{-(\mu-\sigma^2/2)t}\right)$, which is a mean-adjusted nonannualized realized growth rate.

[12] Given our calibration of δ_t process as a random walk, the one additional state variable besides t and D_t is the Radon-Nikodym derivative η_t, or equivalently, $\int_0^t Z_s^2 ds$ (which together with D_t determines η_t). Thus, we plot the averaged values over $\int_0^t Z_s^2 ds$.

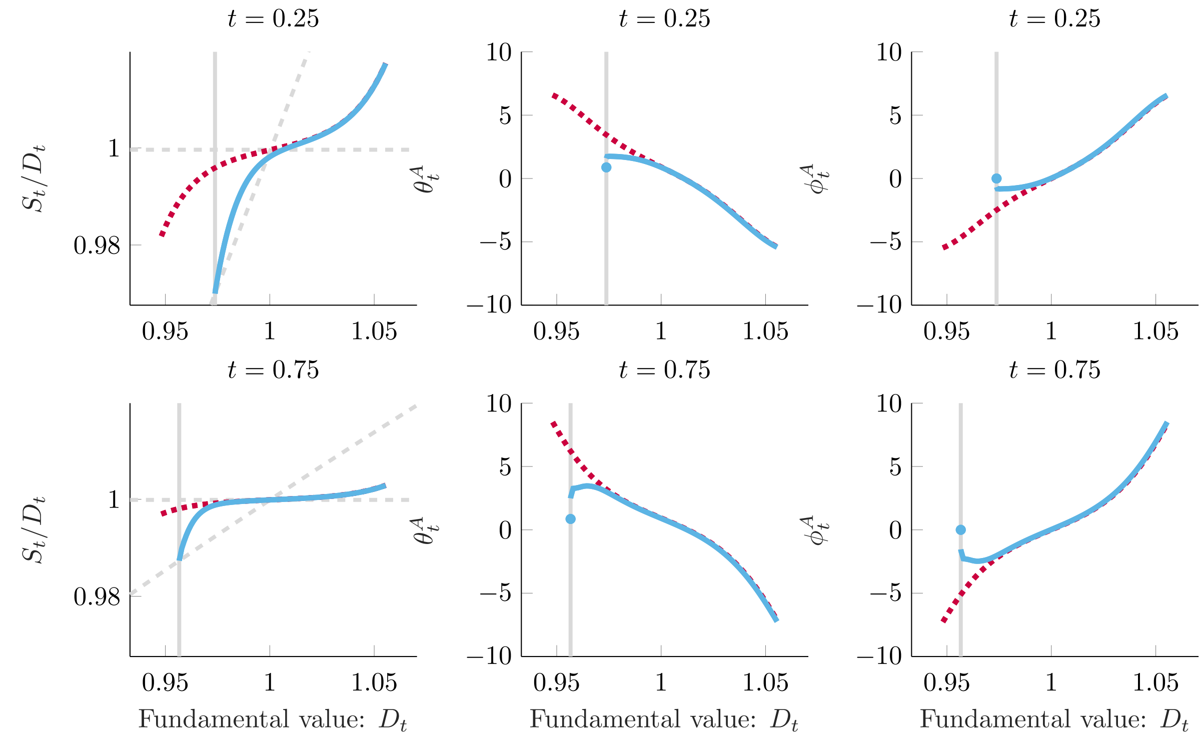

Figure 3: **Price-dividend ratio and agent A's portfolio in the case of time-varying disagreements.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

which is perfectly correlated with D_t, the price-dividend ratio under agent B's beliefs now increases with D_t (the upward-sloping grey dash line).

Next, in the presence of a circuit breaker, for any given level of dividend D_t above the circuit breaker threshold, the price-dividend ratio is again lower than the value without circuit breaker, and the difference becomes more pronounced as D_t approaches the threshold.

However, the circuit breaker does rule out some extreme values for the price-dividend ratio during the trading session, which could occur for really low values of dividend if trading continues. This result could have significant consequences when there are intra-day mark-to-market requirements for some of the market participants, as a narrower range of price-dividend ratio can help lower the chances of forced liquidation when the stock price falls by a large amount. Thus, one of the benefits of having circuit breakers could be to limit the range of price-dividend ratio in this case.

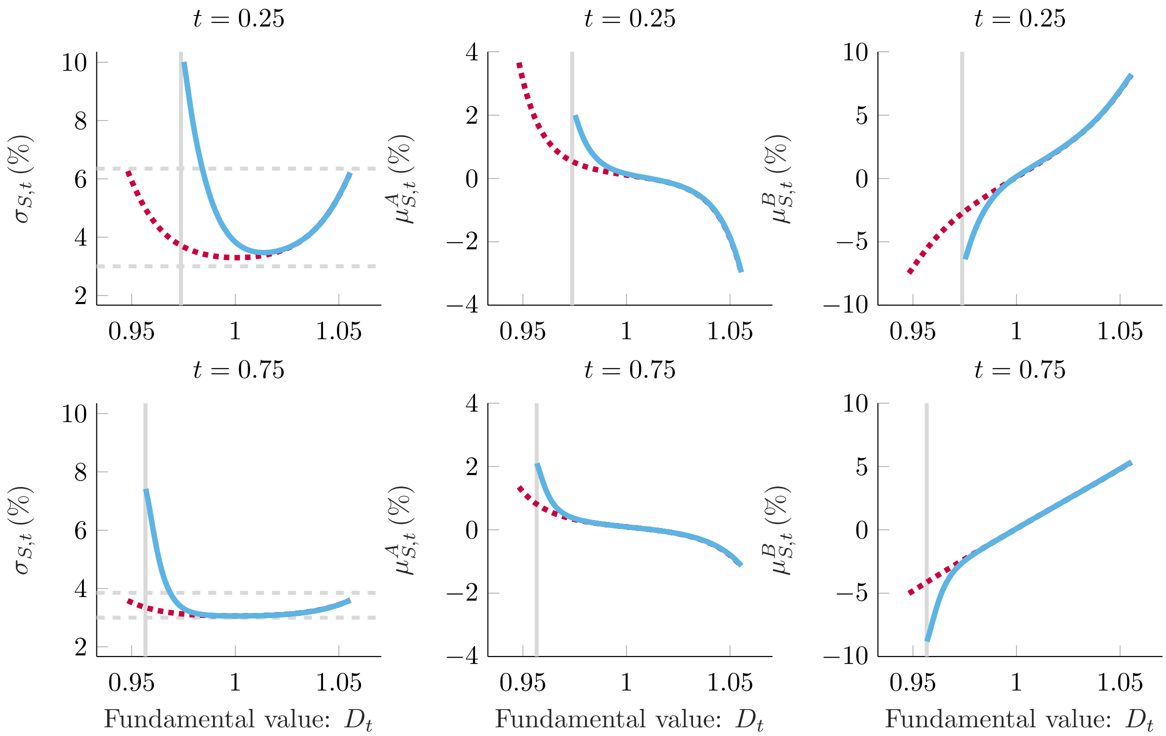

Figure 4: **Conditional volatility and conditional expected returns in the case of time-varying disagreements.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

Similar to the constant disagreement case, circuit breakers in the case of time-varying disagreement amplify the conditional volatility by a significant amount when D_t is close to the threshold $\underline{D}_t$, especially when t is small. A difference is that the conditional volatility is lower with circuit breaker than without when D_t is sufficiently large. As D_t becomes large, so does δ_t, which makes return more volatile. However, the optimistic agent B is made less aggressive by the threat of market closure and loss of liquidity, which is why volatility is dampened in the presence of circuit breakers.

Naturally, the effect of time-varying disagreement becomes more limited as t approaches T. Second, the conditional return volatility now reflects the joint effects of the wealth distribution and the changes in the amount of disagreement. It not only tends to become higher when the distribution of wealth between the two agents is more even, but also when the amount of disagreement is high (large $|\delta_t|$). The result is a U-shaped conditional volatility as function of D_t, which is more pronounced when t is small.

Distributions of price-dividend ratio, daily price range, and realized volatility. Through its impact on the conditional price-dividend ratio, conditional volatility, and the wealth distribution post market closure, circuit breakers also affect the distribution of daily average price-dividend ratios, daily price ranges (defined as daily high minus low prices), and daily return volatilities (the square root of the quadratic variation of $\log(S_t)$ over $[0, T]$). We examine these effects in Figure 5. The top panel shows that the distribution of daily average price-dividend ratio is shifted to the left in the presence of circuit breakers, which shows that circuit breakers indeed lead to more downside price distortion in this model.

Next, the results for the daily price range distribution show that circuit breakers can reduce the probabilities of having very large daily price ranges (those over 6%), but they would raise the probabilities of daily price ranges between 4.5 and 6%. Moreover, circuit breakers generate significant fatter tails for the distributions of daily realized volatilities (in addition to the larger conditional volatilities shown earlier).[13] The results of these two volatility measures both show that the effect of circuit breakers on return volatility is far more intricate than what the naive intuition would suggest.

"The Magnet Effect" "The magnet effect" is a phenomenon that is often associated with circuit breaker and price limits. Informally, it refers to the acceleration of price movement towards the circuit breaker threshold (price limit) as the price approaches the threshold (limit). We try to formalize this notion in our model by considering the conditional probability that the stock price, currently at S_t, will reach the circuit breaker threshold $(1-\alpha)S_0$ within a given period of time h. In the case without circuit breaker, we can again compute the probability of the stock price reaching the same threshold over the period h. As we will show, our version of "magnet effect" refers to the fact that the very presence of circuit breakers increases the probability of the stock price reaching the threshold in a short period of time, with this effect becoming stronger when the stock price is close to the threshold.

In Figure 6, we consider two different horizons, $h = 10$ and 30 minutes. When S_t is

[13] Realized volatility is measured as the square root of the quadratic variation in log price.

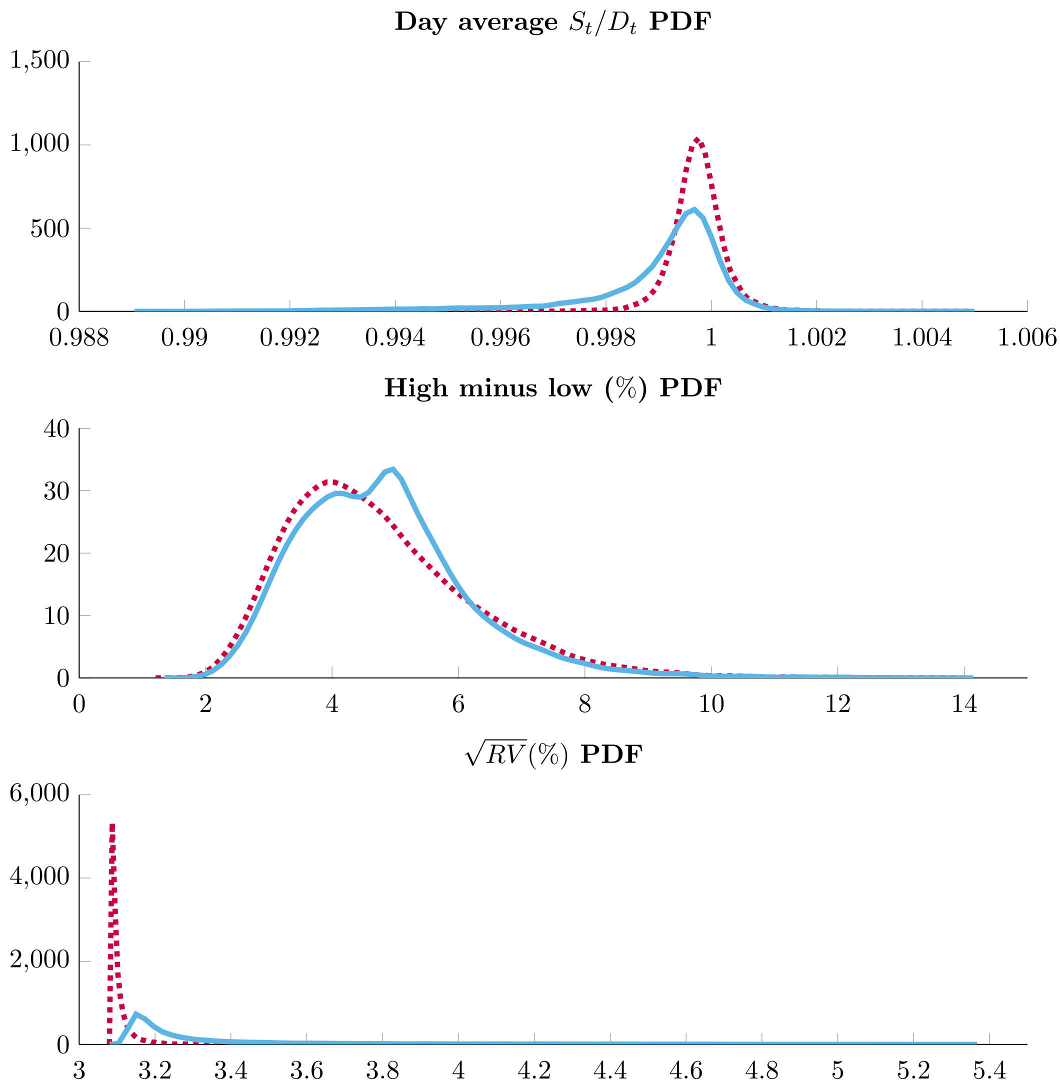

Figure 5: **Distributions of price-dividend ratio, daily price range, and realized volatility.** Solid line: circuit breaker is on; dashed line: complete markets.

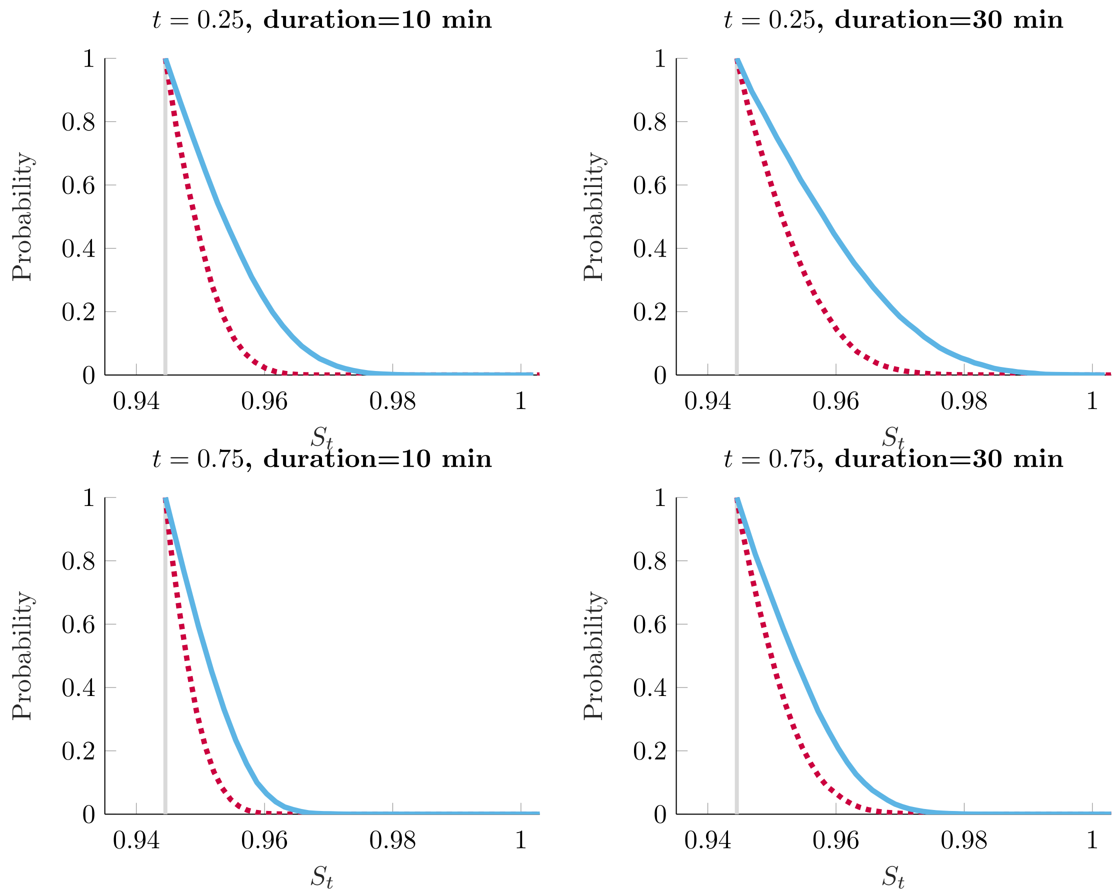

Figure 6: **The "magnet effect".** Probabilities for the stock price to reach the circuit breaker limit within 10 and 30 minutes.

sufficiently far from $(1-\alpha)S_0$, the gap between the conditional probabilities with and without circuit breaker indeed widens as the stock price moves closer to the threshold, which is consistent with the "magnet effect" defined above. This effect is caused by the significant increase in conditional return volatility in the presence of circuit breakers. However, the gap between the two conditional probabilities eventually starts to narrow, because both probabilities will converge to 1 as S_t reaches $(1-\alpha)S_0$. Looking at different horizons h, we see that the largest gap in the two conditional probabilities occurs closer to the threshold when h is small. Moreover, the increase in probability of reaching the threshold is larger earlier during the trading day.

Welfare implications (TO BE COMPLETED)

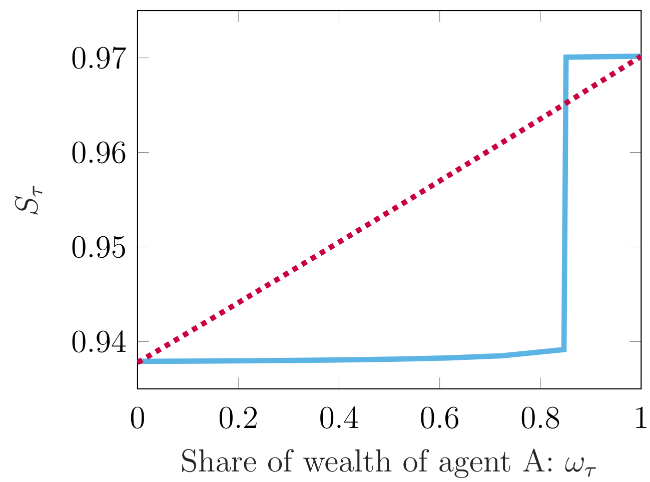

Figure 7: **Stock price upon market closure: positive bond supply.** Blue solid line is for static price. Red dotted line is for the case of complete markets.

4.3 Positive Bond Supply

In this subsection we systematically examine the results when riskless bonds are in positive net supply: $\Delta > 0$. Before presenting the full solution of the model, we first consider the problem at the instant before market closure. This will provide us with much of the intuition of the effect of positive bond supply. Suppose the stock market will close at (an exogenously given) time τ with fundamental value D_τ. We allow trading at time τ, but after that the agents have to hold onto their portfolios until time T. We then calculate equilibrium price of the stock at τ.

In Figure 7, we plot the equilibrium price as a function of the wealth share of agent A for $\tau = 0.25$ and $D_\tau = 0.97$ (the solid line). For comparison, we also plot the price under complete markets (dash lines). Note that for this value of D_τ agent A is relatively more optimistic.

When the optimist's share of total wealth is not too high, the incomplete markets price is lower than its complete markets counterpart, but the opposite occurs when the optimist's wealth share is sufficiently high. The intuition is as follows. As is the case with $\Delta = 0$, the equilibrium price under complete markets is a weighted average of the optimist and pessimist valuations, with the weight depending on their respective wealth shares.

When the optimist's wealth share ω_τ is sufficiently low, he invests 100% of the wealth into the stock and his leverage constraint is binding, as he will not be able to hold all the stocks in the economy. Then the market clearing price has to agree with the pessimist's marginal valuation, since he holds Δ amount of bonds and a positive amount of stocks in his portfolio upon market closure. One difference from the $\Delta = 0$ case is that the valuation of the pessimist increases slowly with ω_τ (instead of remaining constant). This is because the pessimist's portfolio becomes less risky (the fraction of riskless bonds increases relative to the stock) as the optimist becomes wealthier, leading the pessimist to value the stock more. When the optimist's wealth share is sufficiently high, his leverage constraint is no longer binding. He would hold the entire stock market and positive amount of bonds, while the pessimistic agent invests 100% of wealth into bonds. The constraint on the pessimist's short position binds, and the market clearing price has to agree with the optimist's private valuation[14].

The above analysis offers much of the intuition regarding the impact of circuit breakers on the stock price in the setting with positive bond supply. In Figure 8 we draw the heat map for the ratio of averaged realized volatilities in the economy with and without circuit breakers for a wide range of net bond supply (relative to the net supply of stock), as well as a range of initial wealth share for the rational agent A. More precisely, for every value ω and Δ we simulate the economy with and without circuit breakers and calculate average realized volatlity across the sample paths,

$$RV^{CM} = \frac{\mathbb{E}\int_0^T \sigma_t^2 dt}{T}, \tag{44}$$

$$RV^{CB} = \frac{\mathbb{E}\int_0^\tau \sigma_t^2 dt}{\tau}, \tag{45}$$

and draw the heat map for $\sqrt{RV^{CB}}/\sqrt{RV^{CM}} - 1$.

As long as the net bond supply is not too large relative to the size of the equity market, imposing the circuit breakers will have the qualitative effects of raising return

[14]There is also a narrow intermediate range of ω_τ such that both agents are constrained: optimist – by leverage constraint, pessimist – by no short sale. In this case equilibrium price is somewhere in the middle between valuations of the two agents

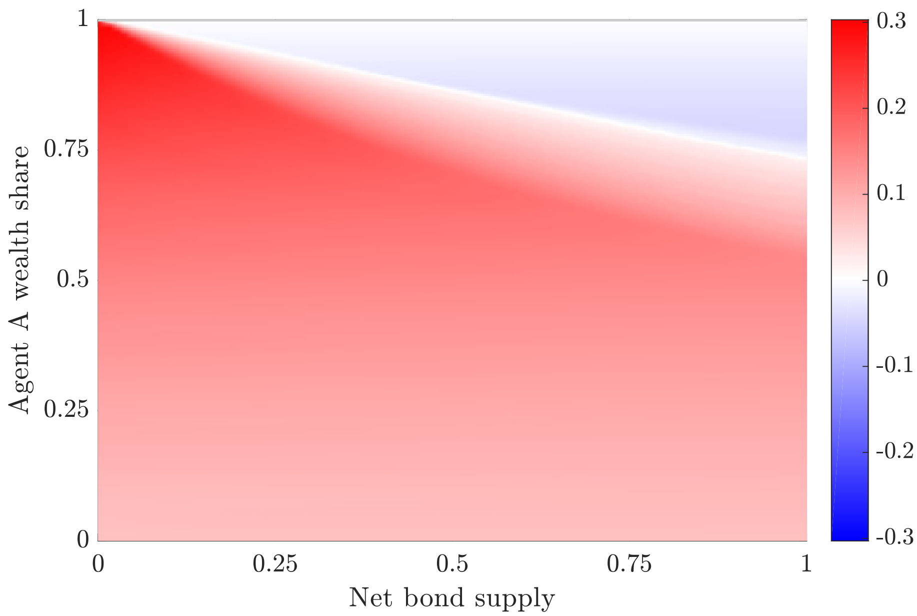

Figure 8: **Heat map: ratio of realized volatilities.** Heat map for the ratio of average realized volatilities in the economies with and without circuit breakers: $\sqrt{RV^{CB}}/\sqrt{RV^{CM}} - 1$.

volatility for a wide range of initial wealth shares (as well as lowering price-dividend ratio). Quantitatively, the volatility amplification is stronger when net bond supply is small, and when agent A's initial wealth share is not too low or too high.

In the data, the net supply of riskless bonds relative to the stock market is likely small. For example, the total size of the U.S. corporate bond market is about \$ 8 trillion in 2016, while the market for equity is about \$23 trillion. If we assume a recovery rate of 50%, the relative size of the market for riskless bonds would be 0.17. Even if one counts the total size of the U.S. market for Treasuries, federal agency securities, and money market instruments (about \$16.8 trillion in 2016) together with corporate bonds, the relative size of the riskless bonds will be less than 1.

5 Robustness

In our model, time is continuous and the dividend follows a geometric Brownian motion. Thus, in a finite time interval Δ, $D_{t+\Delta}$ can in principle take any value on the interval

$(0, \infty)$, as does the stock price $S_{t+\Delta}$, no matter how small Δ is. This feature together with logarithmic utility and the absence of non-equity financial wealth imply that the agents in our model need to completely deleverage and close out any short positions the moment the circuit breaker is triggered to ensure that their wealth remains positive. As discussed above, the complete de-leveraging by the relatively optimistic agents is critical for the pessimists to become the marginal investors upon market closure. This is also why the volatility amplification effect is the strongest when the wealth share of pessimists is approaching zero.

In this section, we examine the robustness of the above results from two perspectives, first by considering shocks to dividend with a finite range, and then by considering the case with non-zero riskless bond supply.

Let's first consider the case where shocks to dividend have a finite range in a given time interval. On the one hand, this means the optimistic (pessimistic) agents will be able to maintain some levered (short) positions during market closures. On the other hand, the size of the cumulated shocks becomes larger over an extended period of market closure, which will put a constraint on the leverage/short positions. One way to capture such a feature is to consider a binomial tree setting. Specifically, we consider a discrete time setting where D_t is modeled using a binomial tree with finite number of steps between 0 and T, and the tree is calibrated to match the mean and volatility of dividend growth of our continuous time model.[15] We consider a tree with 500 steps. As a result, the minimum and maximum values that D_T can take are 0.51 and 1.95 correspondingly.

Before presenting the full solution of this binomial tree model, we first consider the problem at the instant before market closure. Suppose the stock market will close at (an exogenously given) time τ with fundamental value D_τ. We allow trading at time τ, but after that the agents have to hold onto their portfolios until time T. We then calculate equilibrium price of the stock at τ.

In Figure 9, we plot the equilibrium price as a function of the wealth share of agent A (relatively more optimistic agents) for $\tau = 0.25$ and $D_\tau = 0.95$ (the solid lines). For

[15] This economy converges to the continuous time model when the number of steps in the binomial tree goes to infinity.

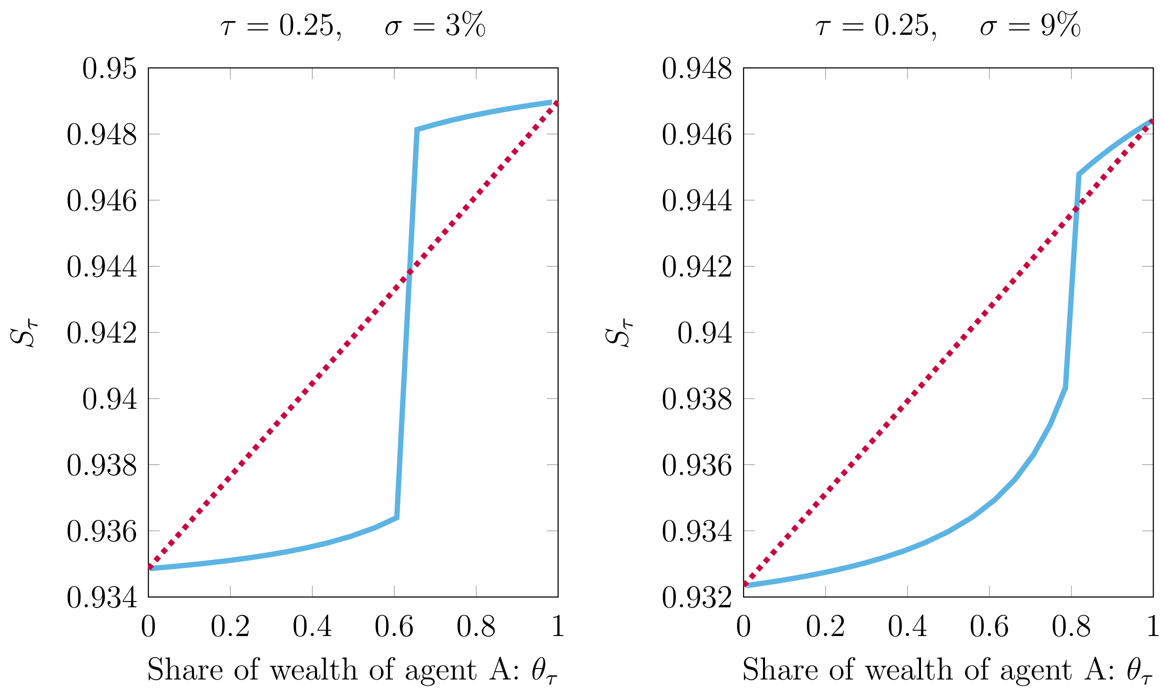

Figure 9: **Stock price upon market closure: binomial tree model.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. Left panel is for the case with dividend volatility $\sigma = 3\%$. Right panel is for the case with $\sigma = 9\%$.

comparison, we also plot the price under complete markets (dash lines). The left panel is for the case with dividend volatility $\sigma = 3\%$, the same as in the continuous time model. The right panel is for the case with higher volatility.

When the optimist's share of total wealth is not too high, the incomplete markets price is lower than its complete markets counterpart, but the opposite occurs when the optimist's wealth share is sufficiently high. The intuition is as follows. As is the case in continuous time, the equilibrium price under complete markets is a weighted average of the optimist and pessimist valuations, with the weight determined by their respective wealth shares. When markets are incomplete, even though agent A will no longer deleverage completely in this binomial model, his leverage is limited by the largest amount the dividend (and price) can drop from τ to T. Similarly, agent B can now take on some short positions, but the short positions are limited by the largest increase in dividend from τ to T.

When the optimist's wealth share ω is sufficiently low, his leverage constraint is binding, as he will not be able to hold all the stocks even at maximum leverage. Then the market clearing price has to agree with the pessimist's private valuation, just as in the continuous time model. One difference from the continuous time model is that the valuation of the pessimist in the binomial tree model increases slowly with ω (instead of remaining constant). This is because the pessimist's portfolio becomes less risky (the fraction of riskless bonds increases relative to the stock) as the optimist becomes wealthier, leading him to value the stock more. Next, when the optimist's wealth share becomes sufficiently high, his leverage constraint is no longer binding. He would hold more than the entire market, while the pessimistic agent takes on a short position. There is an intermediate range of wealth share such that both agents become marginal, as neither is facing a binding constraint (on leverage or short position). Finally, as the optimist's wealth share continues to increase, the constraint on the pessimist's short position eventually binds, and the market clearing price has to agree with the optimist's private valuation.

The above analysis offers much of the intuition regarding the impact of circuit breakers on the stock price in the binomial tree setting. It is easy to see what might move the binomial model closer to the continuous time model in terms of its implication on stock price. Either when the number of steps in the binomial tree or the volatility of the dividend process becomes large, the minimal realization of D_T gets closer to 0. As a result, agent A's leverage constraint in incomplete markets tightens, and the equilibrium price in that case will converge to the pessimist valuation. For example, the effect of high volatility is illustrated in the right panel of Figure 9.

In Figure 10, we plot the price-dividend ratios (left column) and conditional return volatility (right column) as a function of D_t for three different values of optimist wealth share at time $t = 0$: $\theta = 0.3$, $\theta = 0.6$ and $\theta = 0.9$. Red dotted line corresponds to the complete markets case, solid blue line – to the case with circuit breaker on a binomial tree with $n = 500$ steps, dashed orange – to the case with circuit breaker for the baseline model in continuous time (we present it here again for easier comparison). The presented graphs exactly reflect the intuition discussed in the static example. For $\theta = 0.3$ and $\theta = 0.6$ circuit breakers both in discrete and continuous time tend to dampen the prices

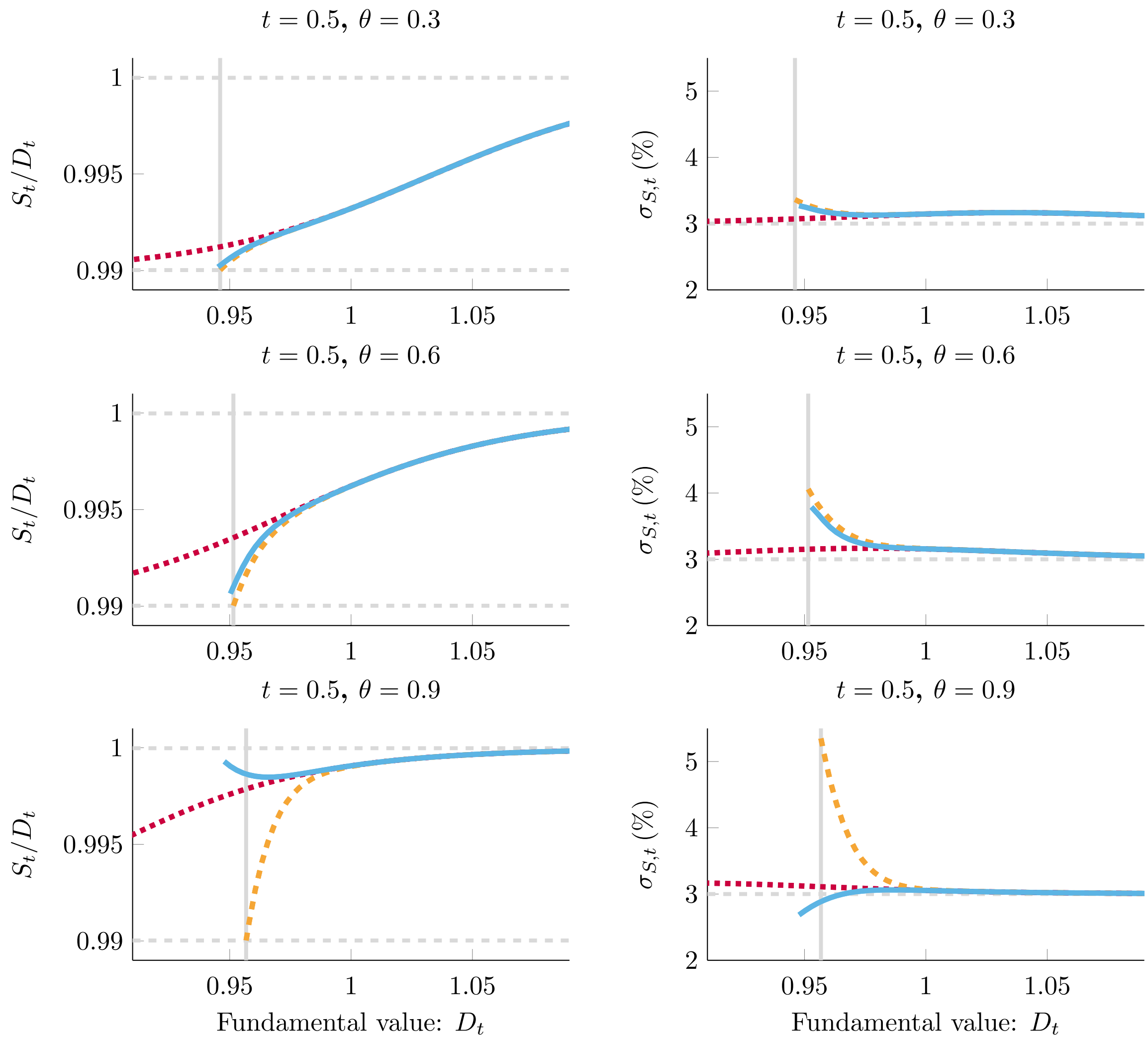

Figure 10: **Price-dividend ratio and conditional volatility.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

and amplify volatility. When $\theta = 0.9$ circuit breakers in the continuous time model and discrete time model have qualitatively opposite effect price level and volatility.

Thus, when the share of wealth owned by the optimist is not too high, circuit breakers will lower price level (relative to complete markets) and increase conditional volatility. The magnitude of the effects is also similar to that in the continuous time model. The opposite happens when the share of wealth owned by the optimist becomes sufficiently high. In that case, circuit breakers can actually inflate prices relative to the complete markets case and reduce return volatility.

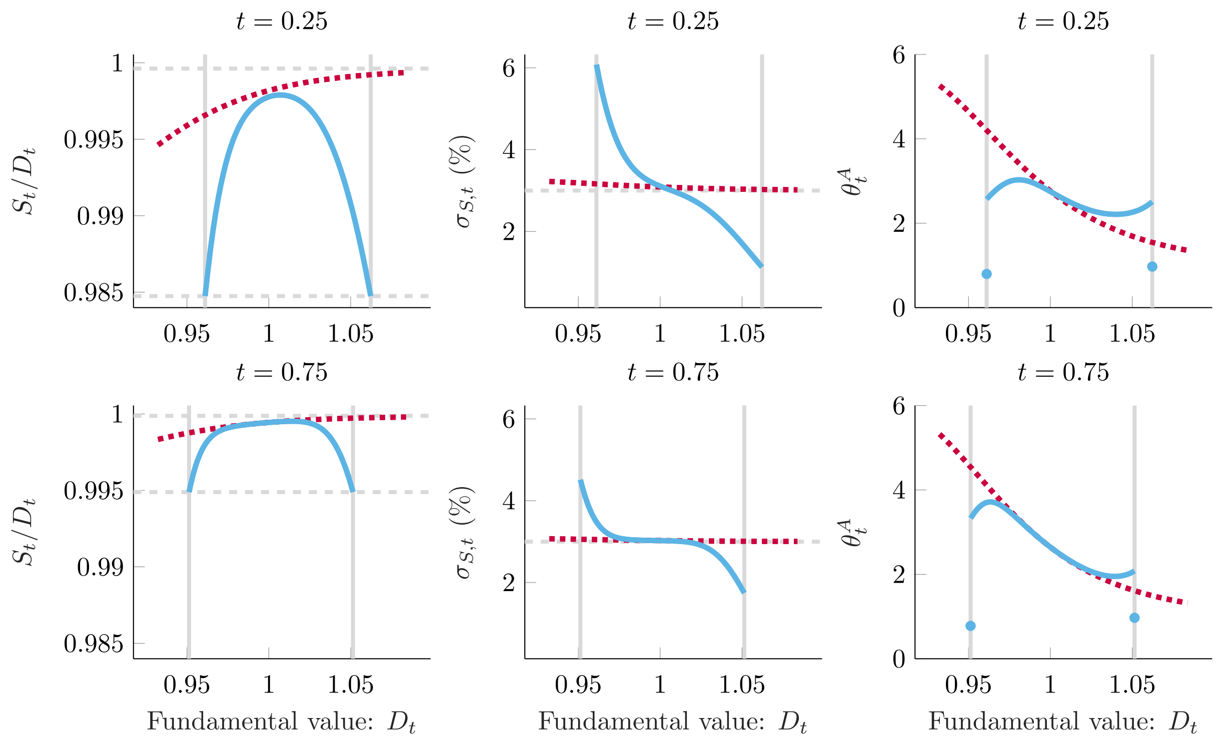

Figure 11: **Price-dividend ratio and agent A's (rational optimist) portfolio holdings.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

Taken together, these results suggest that the findings about the impact of circuit breakers from our benchmark model are robust when the wealth share of the optimistic agents is not too high so that they face binding leverage constraint upon market closure.

6 Extensions

6.1 Two-sided Circuit Breakers

In this section we extend our analysis to the case of the two-sided circuit breakers; i.e. the circuit breaker is triggered when the stock price reaches either $(1-\alpha^D)\,S_0$ or $(1+\alpha^U)\,S_0$ (whatever happens first); $\alpha^U, \alpha^D > 0$. We assume that agents have constant disagreement in beliefs and consider the same numerical parameters as in Section 4.1 with $\alpha^U = \alpha^D = 5\%$.

Figure 11 is the counterpart of Figure 2 for the two-sided circuit breaker case. As before in the left column we present the plots of the price-dividend ratio S_t/D_t as a function of D_t. In the two-sided circuit breaker case the S_t/D_t ratio still lies between the price-dividend ratios $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$ for the two representative-agent economies. Now, however, it has an inverse 'U' shape. For small and intermediary values of D_t price-dividend ratio is an increasing function of D_t. As in one-sided circuit breaker case S_t/D_t equals $\widehat{S}_t^B/D_t$ when $D_t = \underline{D}(t)$ and approaches the price-dividend ratio in the economy without circuit breaker for intermediate values of D_t. For higher values of D_t though, S_t/D_t decreases with D_t and again equals $\widehat{S}_t^B/D_t$ when the dividend reaches the boundary value that triggers the circuit breaker. Interpretation of this behavior of the price-dividend ratio for high values of D_t is similar to the one-sided circuit breaker case. When the circuit breaker is triggered none of the agents is willing to hold either leveraged or short position in stock. In this situation the most pessimistic agent is the marginal investor regardless of whether the stock price has reached the upper or lower boundary.

This behavior of the price-dividend ratio manifests itself in the dependence of conditional volatility on dividend presented in the left column of **??** (the counterpart of the right column of Figure 2 in the one-sided circuit breakers case). Conditional volatility is the highest for low values of D_t when S_t/D_t is increasing in D_t. Conditional volatility gradually decreases with D_t as the slope of S_t/D_t decreases. Finally conditional volatility approaches its lowest value when D_t equals its highest (boundary) value.

This analysis reveals similarities and differences between circuit breakers imposed on the stock price from below and circuit breakers imposed on the stock price from above. In both cases in the presence of circuit breakers price-dividend ratio declines relative to the complete markets economy. The impact on volatility is asymmetric though: lower price boundary tends to amplify volatility, while upper boundary dampens it.

6.2 Multi-level Circuit Breakers

Circuit breakers implemented on exchanges often have more than one price threshold. Intuition we have developed so far in our model can be easily extended to understand the effects of multi-level circuit breakers. Consider for instance our baseline model with zero net bond supply and two price thresholds, α_1 and α_2, $\alpha_1 < \alpha_2$. Whenever the intraday stock price reaches $(1-\alpha_1)S_0$ the trading is stopped for a short period of time δ (e.g. 15 or 30 minutes). If after reopening the price reaches $(1-\alpha_2)S_0$ the trading is stopped till the end of the trading day.

We already know the properties of price dynamics after market reopens following the first trading halt: it is isomorphic to the case of the market with a single-level circuit breaker. To see what happens before the first level is reached consider the agents portfolio choice problem the instant of first market closure, τ_1 (consider the interesting case of $\tau_1 + \delta < T$). Since agents don't have ability to rebalance their portfolios till time $\tau_1 + \delta$ each of them will avoid both leverage and shorting and, hence, will invest 100% of the wealth into the stock market. In that case the pessimist (agent B) has to be the marginal investor that determines stock price. The pessimist's valuation of the stock at time τ_1 will depend on his beliefs regarding the distribution of the equilibrium stock price at time $\tau_1 + \delta$, $S_{\tau_1+\delta}$. Since with positive probability trading will resume at time $\tau_1 + \delta$, it must be the case that $S_{\tau_1+\delta} \geq \widehat{S}^B_{\tau_1+\delta}$ with strict inequality if $\widehat{S}^B_{\tau_1+\delta} > (1-\alpha_2)S_0$. If $\widehat{S}^B_{\tau_1+\delta} \leq (1-\alpha_2)S_0$ the second price threshold will immediately be reached after market reopening and trading will be stopped till the end of the trading day. The above argument implies the following inequality,

$$\widehat{S}^B_{\tau_1} < S_{\tau_1}. \tag{46}$$

To sum up, upon hitting the first threshold each agent will allocate 100% of his wealth into the stock market. The stock price is likely to be dampened relative to the complete markets case. However, different from the single-level circuit breakers, this price will be higher than the pessimist's valuation of the stock in autarky.

7 Conclusion

In this paper, we build a dynamic model to examine the mechanism through which market-wide circuit breakers affect trading and price dynamics in the stock market. As we show, circuit breakers tends to lower the price-dividend ratio, reduce daily price ranges, but increase conditional and realized volatilities. They also raise the probability of the stock price reaching the circuit breaker limit as the price approaches the threshold (a "magnet" effect). The effects of circuit breakers can be further amplified when some agents' willingness to hold the stock is sensitive to recent shocks to fundamentals, which can be due to behavioral biases, institutional constraints, etc. Finally, using historical data from a period when circuit breakers have not been implemented can lead one to underestimate the likelihood of triggering a circuit breaker, especially when the threshold is relatively tight.

Appendix

A Proofs

A.1 Proof of Proposition 1

When there are no circuit breakers, the stock price is

$$\widehat{S}_t = \mathbb{E}_t\left[\frac{\widehat{\pi}_T^A D_T}{\mathbb{E}_t\left[\widehat{\pi}_T^A\right]}\right] = \frac{\mathbb{E}_t\left[\theta + (1-\theta)\,\eta_T\right]}{\mathbb{E}_t\left[D_T^{-1}\left(\theta + (1-\theta)\,\eta_T\right)\right]} = \frac{\theta + (1-\theta)\,\eta_t}{\theta \mathbb{E}_t\left[D_T^{-1}\right] + (1-\theta)\,\mathbb{E}_t\left[D_T^{-1}\eta_T\right]}, \tag{A.1}$$

where

$$\mathbb{E}_t\left[D_T^{-1}\right] = D_t^{-1} e^{-\left(\mu-\sigma^2\right)(T-t)}, \tag{A.2}$$

and

$$\mathbb{E}_t\left[D_T^{-1}\eta_T\right] = \eta_t \mathbb{E}_t\left[D_T^{-1}\frac{\eta_T}{\eta_t}\right] = \eta_t \mathbb{E}_t^B\left[D_T^{-1}\right]. \tag{A.3}$$

Define the log dividend $x_t = \log D_t$. Under measure $\mathbb{P}^B$, the processes for x_t and δ_t are

$$dx_t = \left(\mu - \frac{\sigma^2}{2} + \delta_t\right) dt + \sigma dZ_t^B, \tag{A.4a}$$

$$d\delta_t = \left(\kappa\bar{\delta} + \left(\frac{\nu}{\sigma} - \kappa\right)\delta_t\right) dt + \nu dZ_t^B. \tag{A.4b}$$

Define $X_t = [x_t \;\; \delta_t]'$, then X_t follows an affine process,

$$dX_t = \left(K_0 + K_1 X_t\right) dt + \sigma_X dZ_t^B, \tag{A.5}$$

with

$$K_0 = \begin{bmatrix} \mu - \frac{\sigma^2}{2} \\ \kappa\bar{\delta} \end{bmatrix},\; K_1 = \begin{bmatrix} 0 & 1 \\ 0 & \frac{\nu}{\sigma} - \kappa \end{bmatrix},\; \sigma_X = \begin{bmatrix} \sigma \\ \nu \end{bmatrix}. \tag{A.6}$$

We are interested in computing

$$g\left(t, X_t\right) = \mathbb{E}_t^B\left[e^{\rho_1' X_T}\right], \quad \text{with } \rho_1 = [-1 \;\; 0]'. \tag{A.7}$$

By applying standard results for the conditional moment-generating functions of affine processes (see, .e.g., Singleton 2006), we get

$$g\left(t, X_t\right) = \exp\left(A\left(t,T\right) + B\left(t,T\right)' X_t\right), \tag{A.8}$$

where

$$\begin{aligned} 0 &= \dot{B} + K_1' B, \quad B\left(T,T\right) = \rho_1 && \text{(A.9a)} \\ 0 &= \dot{A} + B' K_0 + \frac{1}{2} tr\left(BB'\sigma_X \sigma_X'\right), \quad A\left(T,T\right) = 0 && \text{(A.9b)} \end{aligned}$$

Solving for the ODEs gives:

$$B(t,T) = \left[-1 \quad \frac{1 - e^{\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}}{\frac{\nu}{\sigma} - \kappa}\right]', \tag{A.10}$$

and

$$\begin{aligned} A(t,T) = &\left[\mu - \sigma^2 - \frac{\kappa\bar{\delta} - \sigma\nu}{\frac{\nu}{\sigma} - \kappa} - \frac{\nu^2}{2\left(\frac{\nu}{\sigma} - \kappa\right)^2}\right](t - T) - \frac{\nu^2}{4\left(\frac{\nu}{\sigma} - \kappa\right)^3}\left[1 - e^{2\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}\right] \\ &+ \left[\frac{\kappa\bar{\delta} - \sigma\nu}{\left(\frac{\nu}{\sigma} - \kappa\right)^2} + \frac{\nu^2}{\left(\frac{\nu}{\sigma} - \kappa\right)^3}\right]\left[1 - e^{\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}\right]. \end{aligned} \tag{A.11}$$

After plugging the above results back into (A.1) and reorganizing the terms, we get

$$S_t = \frac{\theta + (1-\theta)\eta_t}{\theta + (1-\theta)\eta_t H(t, \delta_t)} D_t e^{\left(\mu - \sigma^2\right)(T-t)}, \tag{A.12}$$

where

$$\begin{aligned} H(t, \delta_t) &= e^{a(t,T) + b(t,T)\delta_t}, && \text{(A.13a)} \\ a(t,T) &= \left[\frac{\kappa\bar{\delta} - \sigma\nu}{\frac{\nu}{\sigma} - \kappa} + \frac{\nu^2}{2\left(\frac{\nu}{\sigma} - \kappa\right)^2}\right](T - t) - \frac{\nu^2}{4\left(\frac{\nu}{\sigma} - \kappa\right)^3}\left[1 - e^{2\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}\right] \\ &\quad + \left[\frac{\kappa\bar{\delta} - \sigma\nu}{\left(\frac{\nu}{\sigma} - \kappa\right)^2} + \frac{\nu^2}{\left(\frac{\nu}{\sigma} - \kappa\right)^3}\right]\left[1 - e^{\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}\right], && \text{(A.13b)} \\ b(t,T) &= \frac{1 - e^{\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}}{\frac{\nu}{\sigma} - \kappa}. && \text{(A.13c)} \end{aligned}$$

Finally, to compute the conditional volatility of stock returns, we have

$$\begin{aligned} d\widehat{S}_t &= \widehat{\mu}_{S,t}\widehat{S}_t dt + \widehat{\sigma}_{S,t}\widehat{S} dZ_t \\ &= o(dt) + \widehat{S}_t \frac{dD_t}{D_t} + \eta_t D_t e^{\left(\mu-\sigma^2\right)(T-t)} \frac{\theta(1-\theta)[1-H(t,\delta_t)]}{[\theta+(1-\theta)\eta_t H(t,\delta_t)]^2} \frac{d\eta_t}{\eta_t} \\ & -D_t e^{\left(\mu-\sigma^2\right)(T-t)} \frac{[\theta+(1-\theta)\eta_t](1-\theta)\eta_t H(t,\delta_t) b(t,T)}{[\theta+(1-\theta)\eta_t H(t,\delta_t)]^2} d\delta_t. \end{aligned}$$

After collecting the diffusion terms, we get

$$\begin{aligned} \widehat{\sigma}_{S,t} = \sigma + \frac{D_t e^{\left(\mu-\sigma^2\right)(T-t)}}{\widehat{S}_t} \Bigg\{ & \frac{\theta(1-\theta)\left[1-H(t,\delta_t)\right]}{[\theta+(1-\theta)\eta_t H(t,\delta_t)]^2} \frac{\delta_t \eta_t}{\sigma} \\ & - \frac{\left[\theta+(1-\theta)\eta_t\right](1-\theta)\eta_t b(t,T) H(t,\delta_t)}{[\theta+(1-\theta)\eta_t H(t,\delta_t)]^2} \nu \Bigg\}. \end{aligned} \tag{A.14}$$

A.2 Proof of Proposition 3

Suppose market closes at time τ. Since bonds are zero net supply,

$$W_{1,\tau} + W_{2,\tau} = S_\tau.$$

Then the agents' problems at time τ are:

$$\begin{aligned} V^1\left(W_{1,\tau}\right) \quad = \quad & \max_{\theta_1,b_1} \quad E^1_\tau\left[\ln\left(\theta_1 D_T + b_1\right)\right] \\ & s.t. \qquad \theta_1 S_\tau + b_1 \leq W_{1,\tau} \\ & \qquad\qquad \theta_1 \geq 0,\ \ b_1 \geq 0 \end{aligned}$$

and

$$\begin{aligned} V^2\left(W_{2,\tau}\right) \quad = \quad & \max_{\theta_2,b_2} \quad E^2_\tau\left[\ln\left(\theta_2 D_T + b_2\right)\right] \\ & s.t. \qquad \theta_2 S_\tau + b_2 \leq W_{2,\tau} \\ & \qquad\qquad \theta_2 \geq 0,\ \ b_2 \geq 0. \end{aligned}$$

The Lagrangian:

$$L = E^1_\tau\left[\ln\left(\theta_1 D_T + b_1\right)\right] + \zeta^1\left(W_{1,\tau} - \theta_1 S_\tau - b_1\right) + \xi^1_a \theta_1 + \xi^1_b b_1.$$

FOC:

$$
\begin{aligned}
0 &= E_\tau^i\left[\frac{D_T}{\theta_i D_T + b_i}\right] - \zeta^i S_\tau + \xi_a^i \\
0 &= E_\tau^i\left[\frac{1}{\theta_i D_T + b_i}\right] - \zeta^i + \xi_b^i.
\end{aligned}
$$

Suppose agent 1 is less optimistic than agent 2. Then we can quickly examine following three cases:

1. It could be an equilibrium when the price is sufficiently low such that both agents want to take levered positions (putting more than 100% of their wealth in the stock) but are constrained from borrowing. In this case, both agents submit demands proportional to their wealth:

$$
\begin{aligned}
\theta_i^* &= \frac{W_{i,\tau}}{S_\tau} \\
\xi_a^1 &= \xi_a^2 = 0 \\
\xi_b^1 &> 0, \quad \xi_b^2 > 0
\end{aligned}
$$

and the market for the stock clears. In this case,

$$
S_\tau = \frac{E_\tau^1\left[\frac{D_T}{\theta_1 D_T + b_1}\right]}{E_\tau^1\left[\frac{1}{\theta_1 D_T + b_1}\right] + \xi_b^1} = \frac{1}{E_\tau^1\left[\frac{1}{D_T}\right] + \xi_b^1\theta_1} < \frac{1}{E_\tau^1\left[\frac{1}{D_T}\right]} = S_\tau^*.
$$

2. It could be an equilibrium when agent 1 (pessimist) finds it optimal to hold all his wealth in the stock, while agent 2 (optimist) is constrained from borrowing:

$$
\begin{aligned}
\theta_i^* &= \frac{W_{i,\tau}}{S_\tau} \\
\xi_a^1 &= \xi_a^2 = 0 \\
\xi_b^1 &= 0, \quad \xi_b^2 > 0.
\end{aligned}
$$

Then from agent 1,

$$
\begin{aligned}
0 &= E^1_\tau\left[\frac{D_T}{\theta_1 D_T + b_1}\right] - \zeta^1 S_\tau \\
0 &= E^1_\tau\left[\frac{1}{\theta_1 D_T + b_1}\right] - \zeta^1.
\end{aligned}
$$

This implies:

$$
S_\tau = \frac{E^1_\tau\left[\frac{D_T}{\theta_1^* D_T + b_1^*}\right]}{E^1_\tau\left[\frac{1}{\theta_1^* D_T + b_1^*}\right]} = \frac{E^1_\tau\left[\frac{1}{\theta_1^*}\right]}{E^1_\tau\left[\frac{1}{\theta_1^* D_T}\right]} = \frac{1}{E^1_\tau\left[\frac{1}{D_T}\right]} = D_\tau e^{\left(\mu-\sigma^2\right)(T-\tau)} = S^*_\tau.
$$

The latter follows from the fact that market clearing for bonds implies that $b_1^* = b_2^* = 0$, and $\theta_1^* > 0$ as long as $W_{1,\tau} > 0$. Let's check whether this is this consistent with agent 2:

$$
\begin{aligned}
0 &= E^2_\tau\left[\frac{D_T}{\theta_2 D_T + b_1}\right] - \zeta^2 S_\tau \\
0 &= E^2_\tau\left[\frac{1}{\theta_2 D_T + b_2}\right] - \zeta^2 + \xi_b^2 \\
S_\tau &= \frac{E^2_\tau\left[\frac{D_T}{\theta_2^* D_T + b_2^*}\right]}{E^2_\tau\left[\frac{1}{\theta_2^* D_T + b_2^*}\right] + \xi_b^2} = \frac{E^2_\tau\left[\frac{1}{\theta_2^*}\right]}{E^2_\tau\left[\frac{1}{\theta_2^* D_T}\right] + \xi_b^2} = \frac{1}{E^2_\tau\left[\frac{1}{D_T}\right] + \xi_b^2 \theta_2^*}.
\end{aligned}
$$

Since agent 2 is more optimistic, we have:

$$
E^1_\tau\left[\frac{1}{D_T}\right] > E^2_\tau\left[\frac{1}{D_T}\right],
$$

which implies:

$$
\xi_b^2 = \frac{E^1_\tau\left[\frac{1}{D_T}\right] - E^2_\tau\left[\frac{1}{D_T}\right]}{\theta_2^*} > 0.
$$

3. For any $S_\tau > S^*_\tau$, agent 1 will prefer to hold less than 100% of the wealth in the stock. This would require agent 2 to take levered position, which cannot be an equilibrium.

We will restrict our attention to equilibriums of type 2.

A.3 Special Case: Constant Disagreement

The stock price can be computed in closed form in the case of constant disagreement, $\delta_t = \delta$. Without loss of generality, we focus on the case where agent B is relatively more optimistic,

$\delta \geq 0$. The results are summarized below.

Proposition 4. *Take S_0 as given. With $\delta \geq 0$, the stock price time $t \leq \tau \wedge T$ is*

$$S_t = \left(\omega_t^A \mathbb{E}_t[S_{\tau\wedge T}^{-1}] + \omega_t^B \mathbb{E}_t^B[S_{\tau\wedge T}^{-1}]\right)^{-1}, \tag{A.15}$$

where

$$\begin{aligned}
\mathbb{E}_t[S_{\tau\wedge T}^{-1}] = \frac{1}{\alpha S_0} & \left\{ N\left[\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right] + e^{\sigma d_t} N\left[\frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right]\right\} \\
& + D_t^{-1} e^{-\left(\mu-\sigma^2\right)(T-t)} \left\{ N\left[-\frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right] - e^{-\sigma d_t} N\left[\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right]\right\},
\end{aligned} \tag{A.16}$$

$$\begin{aligned}
\mathbb{E}_t^B[S_{\tau\wedge T}^{-1}] = \frac{1}{\alpha S_0} & \left\{ N\left[\frac{d_t - \left(\frac{\delta}{\sigma} + \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right] + e^{\left(\sigma + \frac{2\delta}{\sigma}\right) d_t} N\left[\frac{d_t + \left(\frac{\delta}{\sigma} + \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right\} \\
& + D_t^{-1} e^{-\left(\mu-\sigma^2+\delta\right)(T-t)} \left\{ N\left[-\frac{d_t - \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right] \right. \\
& \left. - e^{\left(\frac{2\delta}{\sigma} - \sigma\right) d_t} N\left[\frac{d_t + \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right\},
\end{aligned} \tag{A.17}$$

and

$$d_t = \frac{1}{\sigma}\left[\log\left(\frac{\alpha S_0}{D_t}\right) - \left(\mu - \sigma^2\right)(T-t)\right]. \tag{A.18}$$

Proof. As show in Section 3.2, the stock price at time $t \leq \tau \wedge T$ is

$$\begin{aligned}
S_t = \frac{\mathbb{E}_t\left[\pi_{\tau\wedge T}^A S_{\tau\wedge T}\right]}{\pi_t^A} &= \frac{\theta + (1-\theta)\eta_t}{\mathbb{E}_t\left[\frac{\theta + (1-\theta)\eta_{\tau\wedge T}}{S_{\tau\wedge T}}\right]} \\
&= \frac{1}{\frac{\theta}{\theta+(1-\theta)\eta_t}\mathbb{E}_t\left[S_{\tau\wedge T}^{-1}\right] + \frac{(1-\theta)\eta_t}{\theta+(1-\theta)\eta_t}\mathbb{E}_t\left[\frac{\eta_{\tau\wedge T}}{\eta_t} S_{\tau\wedge T}^{-1}\right]} \\
&= \frac{1}{\frac{\theta}{\theta+(1-\theta)\eta_t}\mathbb{E}_t\left[S_{\tau\wedge T}^{-1}\right] + \frac{(1-\theta)\eta_t}{\theta+(1-\theta)\eta_t}\mathbb{E}_t^B\left[S_{\tau\wedge T}^{-1}\right]}.
\end{aligned} \tag{A.19}$$

The second equality follows from Doob's Optional Sampling Theorem, while the last equality follows from Girsanov's Theorem.

Now consider the case when $\delta_t = \delta$. Taking S_0 as given and imposing the condition for stock price at the circuit breaker trigger, we have

$$\mathbb{E}_t\left[S_{\tau\wedge T}^{-1}\right] = \frac{1}{\alpha S_0}P_t\left(\tau\le T\right)+\mathbb{E}_t\left[D_T^{-1}\mathbb{1}_{\{\tau>T\}}\right], \tag{A.20}$$

$$\mathbb{E}_t^B\left[S_{\tau\wedge T}^{-1}\right] = \frac{1}{\alpha S_0\eta_t}\mathbb{E}_t\left[\eta_\tau\mathbb{1}_{\{\tau\le T\}}\right]+\frac{1}{\eta_t}\mathbb{E}_t\left[\eta_T D_T^{-1}\mathbb{1}_{\{\tau>T\}}\right]. \tag{A.21}$$

The following standard results about hitting times of Brownian motions are helpful for deriving the expressions for the expectations in (A.20)-(A.21) (see e.g., Jeanblanc, Yor, and Chesney, 2009, chap 3). Let Z^μ denote a drifted Brownian motion, $Z_t^\mu=\mu t+Z_t$, with $Z_0^\mu=0$. Let $\mathcal{T}_y^\mu=\inf\left\{t\ge 0: Z_t^\mu=y\right\}$ for $y<0$. Then:

$$\Pr\left(T_y^\mu\le t\right) = N\left(\frac{y-\mu t}{\sqrt{t}}\right)+e^{2\mu y}N\left(\frac{y+\mu t}{\sqrt{t}}\right), \tag{A.22}$$

$$E\left[e^{-\lambda T_y^\mu}\mathbb{1}_{\{T_y^\mu\le t\}}\right] = e^{(\mu-\gamma)y}N\left(\frac{y-\gamma t}{\sqrt{t}}\right)+e^{(\mu+\gamma)y}N\left(\frac{y+\gamma t}{\sqrt{t}}\right), \tag{A.23}$$

where $\gamma=\sqrt{2\lambda+\mu^2}$.

Recall the definition of the stopping time τ in Equation (37), which simplifies in the case with constant disagreement,

$$\tau=\inf\left\{t\ge 0: D_t=\alpha S_0 e^{-\left(\mu-\sigma^2\right)(T-t)}\right\}. \tag{A.24}$$

Through a change of variables, we can redefine τ as the first hitting time of a drifted Brownian motion for a constant threshold. Specifically, define:

$$y_t=\frac{1}{\sigma}\log\left(e^{-\left(\mu-\sigma^2\right)t}D_t\right), \tag{A.25}$$

then $y_0=0$, and

$$y_t=Z_t^{\frac{\sigma}{2}}=\frac{\sigma}{2}t+Z_t. \tag{A.26}$$

Moreover,

$$\mathcal{T}_d^{\frac{\sigma}{2}}=\inf\left\{t\ge 0: y_t=d\right\}\overset{a.s.}{=}\tau, \tag{A.27}$$

where the threshold is constant over time,

$$d=\frac{1}{\sigma}\log\left(\alpha S_0 e^{-\left(\mu-\sigma^2\right)T}\right). \tag{A.28}$$

Conditional on y_t and the fact that the circuit breaker has not been triggered up to time t, the result from (A.22) implies

$$P_t(\tau \le T) = P_t\left(\mathcal{T}_{d_t}^{\frac{\sigma}{2}} \le T - t\right) = N\left[\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right] + e^{\sigma d_t} N\left[\frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right], \tag{A.29}$$

where

$$d_t = d - y_t = \frac{1}{\sigma}\left[\log\left(\frac{\alpha S_0}{D_t}\right) - \left(\mu - \sigma^2\right)(T-t)\right]. \tag{A.30}$$

The threshold d_t is normalized with respect to y_t so as to start the drifted Brownian motion $Z^{\frac{\sigma^2}{2}}$ from 0 at time t.

Next,

$$\begin{aligned}
&\mathbb{E}_t\left[D_T^{-1}\mathbb{1}_{\{\tau>T\}}\right] \\
&= D_t^{-1} e^{-\left(\mu - \frac{\sigma^2}{2}\right)(T-t)} \mathbb{E}_t\left[e^{-\sigma(Z_T - Z_t)}\mathbb{1}_{\{\tau>T\}}\right] \\
&= D_t^{-1} e^{-\left(\mu - \sigma^2\right)(T-t)} \mathbb{E}_t\left[e^{-\sigma(Z_T - Z_t) - \frac{\sigma^2}{2}(T-t)}\mathbb{1}_{\{\tau>T\}}\right] \\
&= D_t^{-1} e^{-\left(\mu - \sigma^2\right)(T-t)} \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>T\}}\right] \\
&= D_t^{-1} e^{-\left(\mu - \sigma^2\right)(T-t)} \left\{N\left[-\frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right] - e^{-\sigma d_t} N\left[\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right]\right\}.
\end{aligned} \tag{A.31}$$

The third equality follows from Girsanov's Theorem, and the fourth equality again follows from (A.22). Under $\mathbb{Q}$, $Z_t^{\sigma} = Z_t + \sigma t$ is a standard Brownian motion, and

$$y_t = -\frac{\sigma}{2}t + Z_t^{\sigma}. \tag{A.32}$$

Next, it follows from (A.25) and the definition of τ that

$$y_\tau = y_t + \frac{\sigma}{2}(\tau - t) + (Z_\tau - Z_t) = d. \tag{A.33}$$

Thus,

$$Z_\tau - Z_t = d_t - \frac{\sigma}{2}(\tau - t). \tag{A.34}$$

With these results, we can evaluate the following expectation:

$$\begin{aligned}
\mathbb{E}_t\left[\eta_\tau \mathbb{1}_{\{\tau\leq T\}}\right] &= \mathbb{E}_t\left[\eta_t e^{\frac{\delta}{\sigma}(Z_\tau - Z_t) - \frac{\delta^2}{2\sigma^2}(\tau - t)} \mathbb{1}_{\{\tau\leq T\}}\right] \\
&= \eta_t e^{\frac{\delta d_t}{\sigma}} \mathbb{E}_t\left[\exp\left(-\left(\frac{\delta}{2} + \frac{\delta^2}{2\sigma^2}\right)(\tau - t)\right)\mathbb{1}_{\{\tau\leq T\}}\right] \\
&= \eta_t \left\{N\left[\frac{d_t - \left(\frac{\delta}{\sigma} + \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right] + e^{\left(\sigma + \frac{2\delta}{\sigma}\right)d_t} N\left[\frac{d_t + \left(\frac{\delta}{\sigma} + \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right\},
\end{aligned}$$

where the last equality follows from an application of (A.23).

Finally,

$$\begin{aligned}
\mathbb{E}_t\left[\eta_T D_T^{-1} \mathbb{1}_{\{\tau > T\}}\right] &= \mathbb{E}_t\left[\eta_t e^{\frac{\delta}{\sigma}(Z_T - Z_t) - \frac{\delta^2}{2\sigma^2}(T-t)} D_t^{-1} e^{-\left(\mu - \frac{\sigma^2}{2}\right)(T-t) - \sigma(Z_T - Z_t)} \mathbb{1}_{\{\tau > T\}}\right] \\
&= \eta_t D_t^{-1} e^{-\left(\mu - \sigma^2 + \delta\right)(T-t)} \mathbb{E}_t\left[e^{\left(\frac{\delta}{\sigma} - \sigma\right)(Z_T - Z_t) - \frac{\left(\frac{\delta}{\sigma} - \sigma\right)^2}{2}(T-t)} \mathbb{1}_{\{\tau > T\}}\right] \\
&= \eta_t D_t^{-1} e^{-\left(\mu - \sigma^2 + \delta\right)(T-t)} \mathbb{E}_t^{\widetilde{\mathbb{Q}}}\left[\mathbb{1}_{\{\tau > T\}}\right] \\
&= \eta_t D_t^{-1} e^{-\left(\mu - \sigma^2 + \delta\right)(T-t)} \left\{N\left[-\frac{d_t - \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right. \\
&\quad \left. - e^{\left(\frac{2\delta}{\sigma} - \sigma\right)d_t} N\left[\frac{d_t + \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right\}.
\end{aligned}$$

The third equality follows from Girsanov's Theorem, and the fourth equality follows from (A.22). Under $\widetilde{\mathbb{Q}}$, $Z_t^{\sigma - \frac{\delta}{\sigma}} = Z_t + \left(\sigma - \frac{\delta}{\sigma}\right)t$ is a standard Brownian motion, and

$$y_t = \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right)t + Z_t^{\sigma - \frac{\delta}{\sigma}}. \tag{A.35}$$

□

B Numerical Solution

Now we outline the numerical algorithm used to solve the model for the $\Delta > 0$ case. Time interval $[0, T]$ is discretized using a grid $\{t_0, t_1, ..., t_n\}$, where $t_0 = 0$ and $t_n = T$. For every point t_i on the time grid we construct grids for a set of state variables which uniquely determine fundamental value D_{t_i}, Radon-Nikodym derivative η_{t_i} and disagreement δ_{t_i} [16]. We will denote by $\Theta_{i,k} = (t_i, \Xi_k)$ a tuple (which we will call a "node") that summarizes the state of the economy at time t_i and k here indexes a particular point of the discretized state-space. We assign probabilities of the transition $\Theta_{i,k} \to \Theta_{i+1,j}$ for all i, j, k to match expected value and dispersion of one-step changes in state variables with their continuous time counterparts, namely, drifts and diffusions.

Using this structure we can solve for the equilibrium in complete markets in the following way.

1. For time point $t_n = T$: use equations (8)—(10) to calculate consumption allocations and state-price density in all nodes $\Theta_{n,k}$; set stock price equal to D_T.
2. For time point t_{n-1}: use transition probabilities and the fact that $\pi_t S_t$, $\pi_t \widehat{W}_t^A$, π_t are martingales to calculate the state-price density, stock price and wealth of agent A in nodes $\Theta_{n-1,k}$ for all k.
3. Proceding backwards repeat the above step $n-1$ times to obtain the state-price density, stock price and wealth of agent A in every point $\Theta_{i,k}$.
4. Using transition probabilities calculate drift and diffusion of the stock price process and portfolio holdings of each agent.

In the case with circuit breakers we want to use the algorithm similar to the one above initialized it at time $[\tau \wedge T]$ instead of time T. The problem is that τ is endogenous itself. The following steps show how we solve the problem:

1. In every node $\Theta_{i,k}$: pick a grid for $\omega_{t_i} = W_{t_i}^A/(W_{t_i}^A + W_{t_i}^B)$ spanning the interval $[0, 1]$ and solve the problem (24)—(26) for every point of the grid[17]. Solution to this problem yields "stop" prices $\underline{S}_{i,k,j}$ and marginal utilities of wealth of agents which we will denote

[16] One obvious potential choice of state variables is D_t, η_t and δ_t. However, depending on specific assumptions of the model the state-space dimensionality can be reduced, e.g. in the case of constant disagreement one need to keep track of t and D_t only since they uniquely determine disagreement δ_t and the Radon-Nikodym derivative.

[17] In equilibrium given initial wealth distribution the value $W_{t_i}^A/(W_{t_i}^A + W_{t_i}^B)$ is uniquely pinned down by the Radon-Nikodym derivative η_{t_i} and fundamental value D_{t_i}. Since this relationship is endogenous and is not known before the model is solved our algorithm requires solving the problem for a wide range of wealth distributions.

by $V^{A\prime}_{i,k,j}$ and $V^{B\prime}_{i,k,j}$, where j indexes grid points for ω_{t_i}. Using the planner's problem (28) first order condtion we can define,

$$\lambda_{i,k,j} = \frac{\eta V^{B\prime}_{i,k,j}}{V^{A\prime}_{i,k,j} + \eta V^{B\prime}_{i,k,j}}.$$

2. Now pick an initial guess for the planner's weight λ_g and price threshold $\underline{S}$. In every node $\Theta_{i,k}$ using the values $\lambda_{i,k,j}$ and $S_{i,k,j}$ from the previous step find the "stop" price $\underline{S}_{i,k}$ in point λ_g by interpolation. If $\underline{S}_{i,k} \leq \underline{S}$ then the node $\Theta_{i,k}$ will be either a "stop" node or a node that will never be reached in the equilibrium with circuit breakers. For "stop" nodes we define stock price to be equal to $\underline{S}_{i,k}$ and state-price density to be proportional to $V^{A\prime}_{i,k}$ (which is also obtained by interpolation of $V^{A\prime}_{i,k,j}$ in point λ).

The above procedure effectively defines the stopping time rule τ corresponding to the economy with threshold $\underline{S}$ and planner's weight λ_g. Now we can use the backward procedure described for the case of complete markets to obtain the state-price density, stock price and wealth of every agent in every node $\Theta_{i,k}$. Note that initial wealth share of agent A in this economy will be different from both λ_g and ω. The final step is to find λ_g and $\underline{S}$ so that initial wealth share in the resulting economy is equal to ω and $\underline{S} = (1-\alpha)S_0$. This can be done using the standard bisection method.

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